Chapter 4. Hyperbolic Geometry

Reflections in Circles

4.1 Definition: When L is the line in $\mathbb{R}^2$ through the point $a \in \mathbb{R}^2$ perpendicular to the vector $0 \neq u \in \mathbb{R}^2$, the **reflection** in the line L is the map $F_L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F_L(x) = x - 2\,\mathrm{Proj}_u(x-a) = x - \frac{2(x-a)\cdot u}{|u|^2}\;u.$$

When C is the circle in $\mathbb{R}^2$ centred at the point $a \in \mathbb{R}^2$ of radius $r > 0$, the **reflection** (or **inversion**) in the circle C is the map $F_C : \mathbb{R}^2 \setminus \{a\} \to \mathbb{R}^2 \setminus \{a\}$ given by

$$F_C(x) = a + \frac{r^2}{|x-a|^2}(x-a).$$

4.2 Example: When S is the unit circle $S = \left\{x \in \mathbb{R}^2 \big| |x| = 1\right\}$, we have $F_S(x) = \dfrac{x}{|x|^2}$.

4.3 Note: For any line L we have ${F_L}^2 = I$ and for any circle C we have ${F_C}^2 = I$.

4.4 Note: When C is the circle centred at $a \in \mathbb{R}^2$ of radius $r > 0$ note that for $x \in \mathbb{R}^2 \setminus \{a\}$ we have
(1) $|x-a| < r \iff \big|F_C(x) - a\big| > r$,
(2) $|x-a| > r \iff \big|F_C(x) - a\big| < r$ and
(3) $|x-a| = r \iff \big|F_C(x) - a\big| = r$.
In other words, F_C sends points inside C to points outside C and vice versa, and F_C fixes points on the circle C.

4.5 Note: For $u \in \mathbb{R}^2$, let $T_u : \mathbb{R}^2 \to \mathbb{R}^2$ be the **translation** defined by $T_u(x) = x + u$. For $0 \neq t \in \mathbb{R}$, let $D_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the **dilation** (or **scaling map**) given by $D_t(x) = tx$. Note that when C is the circle of radius $r > 0$ centred at $a \in \mathbb{R}^2$ and S is the unit circle, the reflection F_C is equal to the composite

$$F_C = T_a D_r F_S D_{1/r} T_{-a}$$

because for $x \in \mathbb{R}^2 \setminus \{a\}$ we have

$$T_a D_r F_S D_{1/r} T_{-a}(x) = T_a D_r F_S D_{1/r}(x-a) = T_a D_r F_S\big(\tfrac{1}{r}(x-a)\big) = T_a D_r\Big(\frac{\frac{1}{r}(x-a)}{\frac{1}{r^2}|x-a|^2}\Big)$$
$$= T_a D_r\Big(\frac{r}{|x-a|^2}(x-a)\Big) = T_a\Big(\frac{r^2}{|x-a|^2}(x-a)\Big) = a + \frac{r^2}{|x-a|^2}(x-a) = F_C(x).$$

4.6 Theorem: *(Reflections Preserve Lines and Circles) A reflection in a line or a circle sends lines and circles to lines and circles. When C is the circle centred at $a \in \mathbb{R}^2$ of radius $r > 0$, and $b \in \mathbb{R}^2$ with $b \neq a$ and $c = a + t(b-a)$ with $0 \neq t \in \mathbb{R}$, we have the following.*

(1) Any line through the point a is mapped by F_C to the same line.
(2) The line whose nearest point to a is b is mapped to the circle with diameter $a, F_C(b)$,
(3) The circle with diameter a, b is mapped to the line whose nearest point to a is $F_C(b)$,
(4) The circle with diameter b, c is mapped to the circle with diameter $F(b), F(c)$.

Proof: We prove Parts (2) and (4) and note that (3) follows from (2) because ${F_C}^2 = I$ so that $F_C = {F_C}^{-1}$. To prove Part (2), let L be the line whose nearest point to a is b, and let D be the circle with diameter from a to $F_C(b) = a + \frac{r^2}{|b-a|^2}(b-a)$. For $x \in \mathbb{R}^2 \setminus \{a\}$ and $y = F_C(x) = a + \frac{r^2}{|x-a|^2}(x-a)$ we have

$$x \in L \iff (x-b) \cdot (b-a) = 0$$

and

$$\begin{aligned} y \in D &\iff (y-a) \cdot \big(y - F_C(b)\big) = 0 \\ &\iff \tfrac{r^2}{|x-a|^2}(x-a) \cdot \left(\tfrac{r^2}{|x-a|^2}(x-a) - \tfrac{r^2}{|b-a|^2}(b-a) \right) = 0 \\ &\iff \tfrac{r^4}{|x-a|^2} - \tfrac{r^4}{|x-a|^2|b-a|^2}(x-a) \cdot (b-a) = 0 \\ &\iff |b-a|^2 - (x-a) \cdot (b-a) = 0 \iff (b-a) \cdot (b-a) - (x-a) \cdot (b-a) = 0 \\ &\iff \big((b-a) - (x-a)\big) \cdot (b-a) = 0 \iff (b-x) \cdot (b-a) = 0 \iff x \in L. \end{aligned}$$

To prove Part (4), note that since F_C is equal to the composite $F_C = T_a D_r F_S D_{1/r} T_{-a}$, where S is the unit circle $S = \mathbb{S}^1$, it suffices to prove Part (4) in the case that $C = S$. We need to show that when $0 \neq b \in \mathbb{R}^2$ and $0 \neq t \in \mathbb{R}$, the circle D with diameter from b to tb is mapped by F_S to the circle E with diameter from $F_S(b) = \frac{b}{|b|^2}$ to $F_S(tb) = \frac{tb}{|tb|^2} = \frac{b}{t|b|^2}$. When $x \in \mathbb{R}^2 \setminus \{0\}$ and $y = F_S(x) = \frac{x}{|x|^2}$ we have

$$x \in D \iff (x-b) \cdot (x-tb) = 0 \iff |x|^2 - (1+t) x \cdot b + t|b|^2 = 0$$

and

$$\begin{aligned} y \in E &\iff \big(y - F_S(b)\big) \cdot \big(y - F_S(tb)\big) = 0 \\ &\iff \left(\tfrac{x}{|x|^2} - \tfrac{b}{|b|^2} \right) \cdot \left(\tfrac{x}{|x|^2} - \tfrac{b}{t|b|^2} \right) = 0 \\ &\iff \tfrac{1}{|x|^2} - \left(\tfrac{1}{t} + 1 \right) \tfrac{x \cdot b}{|x|^2 |b|^2} + \tfrac{1}{t|b|^2} = 0 \\ &\iff t|b|^2 - (1+t) x \cdot b + |x|^2 = 0 \\ &\iff x \in D. \end{aligned}$$

4.7 Example: Let C be the circle centred at $a = (2,1)$ of radius $r = \sqrt{10}$. Let L be the line $y = x - 1$, and let M be the line $y = x + 1$. Find the images of the lines L and M under the reflection F_C.

4.8 Example: Let C be the circle centred at $a = (1,2)$ of radius $r = \sqrt{5}$. Let D be the circle with diameter from $(-1,4)$ to $(2,3)$ and let E be the circle $(x-3)^2 + (y-2)^2 = 9$. Find the images of circles D and E under the reflection F_C.

4.9 Theorem: *(Reflections are Conformal) Every reflection in a line or a circle is a conformal map. When C is the circle of radius $r > 0$ centred at $a \in \mathbb{R}^2$, the scaling factor of F_C at the point $x \in \mathbb{R}^2 \setminus \{a\}$ is equal to $\frac{r^2}{|x-a|^2}$.*

Proof: When L is a line in $\mathbb{R}^2$, the reflection F_L is an isometry and hence is a conformal map of scaling factor 1. Change the notation used in the statement of the theorem, and let C be the circle centred at (a, b) of radius $r > 0$. Then

$$\begin{aligned}(u, v) = F_C(x, y) &= (a, b) + \frac{r^2}{\big|(x-a, y-b)\big|^2}\big(x-a, y-b\big)\\ &= \Big(a + \frac{r^2(x-a)}{(x-a)^2+(y-b)^2}\,,\, b + \frac{r^2(y-b)}{(x-a)^2+(y-b)^2}\Big).\end{aligned}$$

Recall that F_C is conformal at (x, y) with scaling factor c when $DF_C{}^T DF_C = c^2 I$. Writing $s = x - a$ and $t = y - b$ we have

$$\begin{aligned}DF_C = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} &= \begin{pmatrix} \frac{r^2\left((x-a)^2+(y-b)^2-2(x-a)^2\right)}{\left((x-a)^2+(y-b)^2\right)^2} & \frac{-2r^2(x-a)(y-b)}{\left((x-a)^2(y-b)^2\right)^2} \\ \frac{-2r^2(x-a)(y-b)}{\left((x-a)^2(y-b)^2\right)^2} & \frac{r^2\left((x-a)^2+(y-b)^2-2(y-b)^2\right)}{\left((x-a)^2+(y-b)^2\right)^2} \end{pmatrix}\\ &= \tfrac{r^2}{(s^2+t^2)^2}\begin{pmatrix} t^2-s^2 & -2st \\ -2st & s^2-t^2 \end{pmatrix}\end{aligned}$$

and so

$$DF_C{}^T DF_C = \tfrac{r^4}{(s^2+t^2)^4}\begin{pmatrix} t^2-s^2 & -2st \\ -2st & s^2-t^2 \end{pmatrix}^2 = \tfrac{r^4}{(s^2+t^2)^4}\begin{pmatrix} s^2+t^2 & 0 \\ 0 & s^2+t^2 \end{pmatrix} = \tfrac{r^4}{(s^2+t^2)^2} I.$$

Thus F_C is conformal at (x, y) with scaling factor $\frac{r^2}{s^2+t^2} = \frac{r^2}{(x-a)^2+(y-b)^2}$. Reverting to the notation used in the statement of the theorem, F_C is conformal at x of scaling factor $\frac{r^2}{|x-a|^2}$.

The Poincaré Disc Model of the Hyperbolic Plane

4.10 Definition: We define the **hyperbolic plane** (also called the **Poincaré Disc**) to be the set

$$\mathbb{H}^2 = \big\{x \in \mathbb{R}^2 \big| |x| < 1\big\} = \big\{(x,y) \in \mathbb{R}^2 \big| x^2 + y^2 < 1\big\}.$$

The boundary of $\mathbb{H}^2$ is the **unit circle**

$$\mathbb{S}^1 = \big\{x \in \mathbb{R}^2 \big| |x| = 1\big\} = \big\{(x,y) \in \mathbb{R}^2 \big| x^2 + y^2 = 1\big\}.$$

and points in $\mathbb{S}^1$ are called **points at infinity** or **asymptotic points**. In the hyperbolic plane, (hyperbolic) **length and area** are measured infinitesimally in terms of Euclidean length and area, by

$$d_H L = \frac{2}{1 - |x|^2}\, d_E L \ , \ d_H A = \frac{4}{(1 - |x|^2)^2}\, d_E A.$$

This means that the (hyperbolic) **length** of a curve given parametrically by $x = \alpha(t)$ where $\alpha : [a,b] \to \mathbb{H}^2$ is given by

$$L = \int_{t=a}^{b} d_H L = \int_{t=a}^{b} \frac{2}{1 - |x|^2}\, d_E L = \int_{t=a}^{b} \frac{2}{1 - |\alpha(t)|^2}\, |\alpha'(t)|\, dt$$

and the (hyperbolic) **area** of a region given in Cartesian coordinates by $x \in D \subseteq \mathbb{H}^2$ or by $(x,y) \in D \subseteq \mathbb{H}^2$, or in polar coordinates by $(r,\theta) \in R$, is equal to

$$\begin{aligned} A &= \iint_D d_H A = \iint_{x \in D} \frac{4}{(1 - |x|^2)^2}\, d_E A \\ &= \iint_{(x,y) \in D} \frac{4}{(1 - x^2 - y^2)^2}\, dx\, dy = \iint_{(r,\theta) \in R} \frac{4r}{(1 - r^2)^2}\, dr\, d\theta. \end{aligned}$$

The (hyperbolic) **angle** between two curves at a point in $\mathbb{H}^2$ and the (hyperbolic) **oriented angle** from one curve to another at a point in $\mathbb{H}^2$ are the same as the Euclidean angle between the two curves and the Euclidean oriented angle from one curve to the other.

4.11 Example: Let $u \in \mathbb{H}^2$. Find the hyperbolic length of the line segment from 0 to u.

Solution: The line segment is given by $x = \alpha(t) = tu$ for $0 \le t \le 1$ and we have $\alpha'(t) = u$, so the hyperbolic length is

$$\begin{aligned} L &= \int_{t=0}^{1} \frac{2\,|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt = \int_{t=0}^{1} \frac{2\,|u|}{1 - |tu|^2}\, dt = \int_{t=0}^{1} \frac{2\,|u|}{1 - t^2|u|^2}\, dt \\ &= \int_{t=0}^{1} \frac{|u|}{1 + |u|\,t} + \frac{|u|}{1 - |u|\,t}\, dt = \left[\ln \frac{1 + |u|\,t}{1 - |u|\,t} \right]_{t=0}^{1} = \ln \frac{1 + |u|}{1 - |u|}. \end{aligned}$$

4.12 Example: Find the hyperbolic area of the disc $\big\{x \in \mathbb{H}^2 \big| |x| \le a\big\}$.

Solution: The hyperbolic area is

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{4r}{(1 - r^2)^2}\, dr\, d\theta = \int_{\theta=0}^{2\pi} \left[\frac{2}{1 - r^2} \right]_{r=0}^{a} d\theta = 2\pi \Big(\frac{2}{1 - a^2} - 2 \Big) = \frac{4\pi\, a^2}{1 - a^2}.$$

4.13 Definition: A (hyperbolic) **line** in $\mathbb{H}^2$ is set which is either of the form $L = M \cap \mathbb{H}^2$ for some line M in $\mathbb{R}^2$ through 0, or of the form $L = C \cap \mathbb{H}^2$ for some circle C in $\mathbb{R}^2$ which intersects the unit circle $\mathbb{S}^1$ orthogonally.

4.14 Note: Let C be the circle centred at $a \in \mathbb{R}^2$ of radius $r > 0$. The circle C intersects $\mathbb{S}^1$ orthogonally at b when the radius of $\mathbb{S}^1$ from 0 to b is perpendicular to the radius of C from a to b. This occurs when a lies outside $\mathbb{S}^1$ and the triangle $0, a, b$ is a right-angled triangle with hypotenuse of length $|a|$ and legs of length r and 1. Thus C intersects $\mathbb{S}^1$ orthogonally when $|a| > 1$ and $r^2 = |a|^2 - 1$.

4.15 Note: Let C be a circle which intersects $\mathbb{S}^1$ orthogonally. Since F_C sends $\mathbb{S}^1$ to a circle, and F_C fixes the two intersection points in $C \cap \mathbb{S}^1$, and F_C preserves the right angles at the two intersection points, we see that F_C sends $\mathbb{S}^1$ to itself. It follows that the map F_C restricts to a bijection $F_C : \mathbb{H}^2 \to \mathbb{H}^2$.

4.16 Definition: When $L = M \cap \mathbb{H}^2$ where M is a line through 0 in $\mathbb{R}^2$, we define the **reflection** in L to be the bijective map $F_L = F_M : \mathbb{H}^2 \to \mathbb{H}^2$. When $L = C \cap \mathbb{H}^2$ where C is a circle in $\mathbb{R}^2$ which intersects $\mathbb{S}^1$ orthogonally, we define the **reflection** in L to be the bijective map $F_L = F_C : \mathbb{H}^2 \to \mathbb{H}^2$.

4.17 Theorem: *(Reflections are Isometries) Let L be a line in $\mathbb{H}^2$. Then $F_L : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry on $\mathbb{H}^2$.*

Proof: This is (reasonably) clear when L is given by a line through the origin. Suppose that $L = C \cap \mathbb{H}^2$ where C is the circle centred at $a \in \mathbb{R}^2$ of radius $r = |a|^2 - 1$. The scaling factor of the map F_L at a point x is equal to $\frac{r^2}{|x-a|^2} = \frac{|a|^2-1}{|x-a|^2}$. In order for the map $F_L : \mathbb{H}^2 \to \mathbb{H}^2$ to be an isometry, this scaling factor must exactly compensate for the change in the scaling factor from the point x to the point $F_L(x)$ in the infinitesimal element of hyperbolic arclength, so we need to prove that when $y = F_L(x)$

$$\frac{|a|^2 - 1}{|x-a|^2} = \frac{1 - |y|^2}{1 - |x|^2}.$$

Let $x \in \mathbb{H}^2$ so that $|x| < 1$, and let $y = F_L(x) = a + \frac{r^2}{|x-a|^2}(x-a) = a + \frac{|a|^2-1}{|x-a|^2}(x-a)$. Then

$$\begin{aligned} 1 - |y|^2 &= 1 - \left(a + \tfrac{|a^2|-1}{|x-a|^2}(x-a)\right) \cdot \left(a + \tfrac{|a^2|-1}{|x-a|^2}(x-a)\right) \\ &= 1 - |a|^2 - 2\,\tfrac{|a|^2-1}{|x-a|^2}\, a \cdot (x-a) - \tfrac{(|a|^2-1)^2}{|x-a|^4}|x-a|^2 \\ &= \tfrac{|a|^2-1}{|x-a|^2}\left(-|x-a|^2 - 2\,a \cdot (x-a) - (|a|^2-1)\right) \\ &= \tfrac{|a|^2-1}{|x-a|^2}\left(-|x|^2 + 2\,x \cdot a - |a|^2 - 2\,a \cdot x + 2\,|a|^2 - |a|^2 + 1\right) \\ &= \tfrac{|a|^2-1}{|x-a|^2}\left(1 - |x|^2\right), \end{aligned}$$

as required.

4.18 Theorem: *Given $u, v \in \mathbb{H}^2 \cup \mathbb{S}^1$ with $u \neq v$. there is a unique line L in $\mathbb{H}^2$ which contains (or, in the case of points at infinity, is asymptotic to) u and v.*

Proof: Let $p \in \mathbb{R}^2$ and let C be the circle in $\mathbb{R}^2$ centred at p with radius $r = \sqrt{|p|^2 - 1}$ so that C intersects $\mathbb{S}^1$ orthogonally. Note that

$$u \in C \iff |u-p|^2 = r^2 \iff |u|^2 - 2\,p \cdot u + |p|^2 = |p|^2 - 1 \iff p \cdot u = \tfrac{|u|^2+1}{2}.$$

Similarly, we have $v \in C \iff p \cdot v = \frac{|v|^2+1}{2}$. When $\{u, v\}$ is linearly independent, there is no line in $\mathbb{R}^2$ through 0 which passes through u and v and there is a unique point p such that the above circle C passes through u and v, (namely the point of intersection of the two lines $x \cdot u = \frac{|u|^2+1}{2}$ and $x \cdot v = \frac{|v|^2+1}{2}$, which are are not parallel). Suppose that $\{u, v\}$ is linearly dependent, say $u \neq 0$ and $v = tu$ with $t \in \mathbb{R}$. Then there is a unique line in $\mathbb{R}^2$ through 0 which passes through u and v, namely the line $L = \text{Span}\{u\}$, but we claim that the above circle C cannot pass through both u and v. Suppose, for a contradiction, that $u, v \in C$. Then

$$\frac{t^2|u|^2+1}{2} = \frac{|tu|^2+1}{2} = \frac{|v|^2+1}{2} = p \cdot v = p \cdot (tu) = t(p \cdot u) = t\,\frac{|u|^2+1}{2}$$

so that

$$0 = t^2|u|^2 - t\big(|u|^2+1\big) + 1 = \big(|u|^2 t - 1\big)(t-1)$$

so either $t = 1$ or $t = \frac{1}{|u|^2}$. But if $t = 1$ then $v = tu = u$ and if $t = \frac{1}{|u|^2}$ then $v = tu = \frac{u}{|u|^2}$ so that $|v| = \frac{1}{|u|} > 1$ in which case $v \notin \mathbb{H}^2$.

4.19 Definition: When two hyperbolic lines meet at a point in $\mathbb{H}^2$, we say the lines **intersect**, when two lines are asymptotic at a point in $\mathbb{S}^1$, we say the lines are **asymptotic** (or **critically parallel**), and when two lines do not intersect and are not asymptotic, we say they are **parallel** (or **ultraparallel**).

4.20 Theorem: *(Perpendicular Bisector) Given $u, v \in \mathbb{H}^2$ with $u \neq v$, there is a unique line L in $\mathbb{H}^2$, called the* **perpendicular bisector** *of u and v, for which $F_L(u) = v$.*

Proof: Let $p \in \mathbb{R}^2$ with $|p| > 1$ and let C be the circle in $\mathbb{R}^2$ centred at p with radius $r = \sqrt{|p|^2 - 1}$. In order to have $F_C(u) = v$, the points u and v must lie on the same ray from p, so we must have $p = u + t(v-u)$ for some $t \in \mathbb{R}$ with $t \notin [0,1]$. When $p = u + t(v-u)$with $t \notin [0,1]$ we have

$$\begin{aligned}
F_C(u) = v \iff & |u-p||v-p| = r^2 \\
\iff & \big|-t(v-u)\big|\big|(1-t)(v-u)\big| = |p|^2 - 1 \\
\iff & |t^2 - t||v-u|^2 = \big|u + t(v-u)\big|^2 - 1 \\
\iff & (t^2 - t)|v-u|^2 = |u|^2 + 2t\,u \cdot (v-u) + t^2|v-u|^2 - 1 \\
\iff & -t|v-u|^2 = |u|^2 + 2t\,u \cdot (v-u) - 1 \\
\iff & -t|v|^2 + 2t\,u \cdot v - t|u|^2 = |u|^2 + 2t\,u \cdot v - 2t\,|u|^2 - 1 \\
\iff & t\big(|u|^2 - |v|^2\big) = |u|^2 - 1 \\
\iff & |u| \neq |v| \ \text{ and } \ t = \tfrac{1-|u|^2}{|v|^2 - |u|^2}.
\end{aligned}$$

Also note that the unique line M in $\mathbb{R}^2$ for which $F_M(u) = v$ is the perpendicular bisector of u and v. When $|u| = |v|$ there is no point p for which $F_C(u) = v$, but the perpendicular bisector M passes through 0. When $|u| \neq |v|$, there is a unique point p for which $F_C(u) = v$, namely the point $p = u + t(v-u)$ with $t = \frac{1-|u|^2}{|v|^2-|u|^2}$, and the perpendicular bisector M does not pass through 0.

4.21 Example: In the case that $v = 0$, in the above proof we have $t = \frac{1-|u|^2}{|v|^2-|u|^2} = -\frac{1-|u|^2}{|u|^2}$ and $p = u + t(v-u) = u - tu = (1-t)u = \big(1 + \frac{1-|u|^2}{|u|^2}\big)u = \frac{u}{|u|^2}$. Thus the unique line L in $\mathbb{H}^2$ for which $F_L(u) = 0$ (or equivalently for which $F_L(0) = u$) is the line $L = C \cap \mathbb{H}^2$ where C is the circle centred at $p = \frac{u}{|u|^2}$ of radius $r = \sqrt{|p|^2 - 1}$.

4.22 Remark: Given $u, v \in \mathbb{H}^2$ with $u \neq v$, if L is the line through u and v and M is the perpendicular bisector of u and v, then, because $F_M(u) = v$ and $F_M(v) = u$, it follows that F_M sends the line L to itself, and so the lines L and M intersect orthogonally.

Geodesics and Distance

4.23 Definition: A **geodesic** in $\mathbb{H}^2$ is a smooth curve in $\mathbb{H}^2$ which minimizes the hyperbolic arclength between any two points on the curve.

4.24 Theorem: *The geodesics in $\mathbb{H}^2$ are the hyperbolic lines.*

Proof: First consider a curve C from 0 to u in $\mathbb{H}$. Represent C in polar coordinates by $x = \big(r(t)\cos\theta(t), r(t)\sin\theta(t)\big)$ where $r(t)$ and $\theta(t)$ are smooth functions with $r(t) \geq 0$ for all $t \in [0,1]$. Note that $|\alpha(t)|^2 = r(t)^2$ and we have

$$\begin{aligned}
\alpha'(t) &= \big(r'\cos\theta - r\,\sin\theta\cdot\theta'\,,\; r'\sin\theta + r\,\cos\theta\cdot\theta'\big)\\
|\alpha'(t)|^2 &= (r'\cos\theta)^2 - 2\,rr'\,\sin\theta\cos\theta\cdot\theta' + (r\sin\theta\cdot\theta')^2\\
&\quad + (r'\sin\theta)^2 + 2\,rr'\sin\theta\cos\theta\cdot\theta' + (r\,\cos\theta\cdot\theta')^2\\
&= (r')^2 + (r\theta')^2
\end{aligned}$$

and so the length of C from 0 to u is

$$\begin{aligned}
L(C) &= \int_0^1 \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt = \int_{t=0}^1 \frac{2\sqrt{r'(t)^2 + r(t)^2\theta'(t)^2}}{1-r(t)^2}\,dt\\
&\geq \int_{t=0}^1 \frac{2\sqrt{r'(t)^2}}{1-r(t)^2}\,dt = \int_{t=0}^1 \frac{2|r'(t)|}{1-r(t)^2}\,dt \geq \int_{t=0}^1 \frac{2\,r(t)}{1-r(t)^2}\,dt\\
&= \int_{s=0}^{|u|} \frac{2\,ds}{1-s^2} = \int_{s=0}^{|u|} \frac{1}{1+s} + \frac{1}{1-s}\;ds = \left[\ln\frac{1+s}{1-s}\right]_{s=0}^{|u|} = \ln\frac{1+|u|}{1-|u|}.
\end{aligned}$$

By comparing this with the result from Example 4.11, we see that the length of C is greater than or equal to the length of the straight line segment from 0 to u. Furthermore, the two inequalities in the above calculation only become equalities in the case that $r(t)\theta'(t) = 0$ and $r'(t) \geq 0$ for all t. When this happens we have $\theta'(t) = 0$ whenever $r(t) > 0$, and so θ is constant for $r > 0$, which implies that C is the straight line segment from 0 to u.

Now consider a smooth curve C from u to v in $\mathbb{H}^2$. Let L be the line such that $F_L(0) = u$. Since ${F_L}^2 = I$ we also have $F_L(u) = 0$. Use the isometry F_L to move the curve C to the curve $D = F_L(C)$ from $0 = F_L(u)$ to $w = F_L(v)$. Let M be the straight line from 0 to w and let $N = F_L(M)$. Notice that N is the unique hyperbolic line through u and v. The hyperbolic arclength along C from u to v is equal to the hyperbolic arclength along $D = F_L(C)$ from 0 to w which is greater than or equal to the hyperbolic arclength along the straight line M from 0 to w, which is equal to the hyperbolic arclength along N from u to v. It follows that the hyperbolic line N through u and v is the geodesic from u to v.

4.25 Definition: For $u, v \in \mathbb{H}$ we define the (hyperbolic) **distance** between u and v, denoted by $d_H(u,v)$, to be equal to the hyperbolic arclength along the (unique) line from u to v. The hyperbolic **line segment** between u and v, that is the arc between u and v along the hyperbolic line through u and v, is denoted by $[u,v]$ (we do not normally distinguish notationally between the Euclidean line segment $[u,v]$ and the hyperbolic line segment $[u,v]$, so it is sometimes necessary to specify).

4.26 Theorem: *Let $u, v \in \mathbb{H}^2$. Then*

$$d_H(u,v) = \cosh^{-1}\left(1 + \frac{2|v-u|^2}{(1-|u|^2)(1-|v|^2)}\right).$$

Proof: Let L be the line in $\mathbb{H}^2$ such that $F_L(u) = 0$ and let $w = F_L(v)$. Since F_L is an isometry, we have

$$d_H(u,v) = d_H\big(F_L(u), F_L(v)\big) = d_H(0,w).$$

From Example 4.11, we have $d_H(0,w) = \ln \frac{1+|w|}{1-|w|}$, and so

$$\begin{aligned}\cosh_H d_H(u,v) = \cosh d_H(0,w) = \cosh\left(\ln \frac{1+|w|}{1-|w|}\right) &= \frac{1}{2}\left(\frac{1+|w|}{1-|w|} + \frac{1-|w|}{1+|w|}\right)\\ &= \frac{1}{2}\left(\frac{(1+|w|)^2 + (1-|w|)^2}{1-|w|^2}\right) = \frac{1+|w|^2}{1-|w|^2} = 1 + \frac{2|w|^2}{1-|w|^2}.\end{aligned}$$

Thus to prove the theorem, it suffices to prove that

$$1 + \frac{2|w|^2}{1-|w|^2} = 1 + \frac{2|u-v|^2}{(1-|u|^2)(1-|v|^2)}.$$

Recall from Example 4.20 that $L = C \cap \mathbb{H}^2$ where C is the circle centred at $p = \frac{u}{|u|^2}$ of radius $r = \sqrt{|p|^2 - 1}$. Recall from Theorem 4.9 that F_L is conformal and scales by the factor $\frac{|p|^2-1}{|v-p|^2}$ when it sends v to w. Recall from Theorem 4.17 that F_L is an isometry because this scaling factor compensates for the change in scaling factor from v to w in the definition of $d_H L$ so we have $\frac{|p|^2-1}{|v-p|^2} = \frac{1-|w|^2}{1-|v|^2}$. Thus we have

$$\begin{aligned}1 - |w|^2 &= \frac{|p|^2-1}{|v-p|^2}(1-|v|^2) = \frac{\frac{1}{|u|^2} - 1}{|v-p|^2}(1-|v|^2) = \frac{(1-|u|^2)(1-|v|^2)}{|u|^2|v-p|^2}\\ |w|^2 &= 1 - \big(1-|w|^2\big) = 1 - \frac{(1-|u|^2)(1-|v|^2)}{|u|^2|v-p|^2} \text{ and}\\ &= \frac{|u|^2|v-p|^2 - (1-|u|^2)(1-|v|^2)}{|u|^2|v-p|^2} = \frac{\left||u|v - \frac{u}{|u|}\right|^2 - (1-|u|^2)(1-|v|^2)}{|u|^2|v-p|^2}\\ &= \frac{\big(|u|^2|v|^2 - 2(u \cdot v) + 1\big) - \big(1 - |u|^2 - |v|^2 + |u|^2|v|^2\big)}{|u|^2|v-p|^2}\\ &= \frac{|u|^2 - 2\,u \cdot v + |v|^2}{|u|^2|v-p|^2} = \frac{|u-v|^2}{|u|^2|v-p|^2}\end{aligned}$$

so that $\dfrac{|w|^2}{1-|w|^2} = \dfrac{|u-v|^2}{(1-|u|^2)(1-|v|^2)}$, as required.

4.27 Definition: For $u \in \mathbb{H}^2$ and $r > 0$, the (hyperbolic) **circle** centred at u of radius r and the (hyperbolic) **disc** centred at u of radius r are the sets

$$C_H(u,r) = \{x \in \mathbb{H}^2 \big| d_H(x,u) = r\},$$
$$D_H(u,r) = \{x \in \mathbb{H}^2 \big| d_H(x,u) \leq r\}$$

4.28 Note: Every hyperbolic circle is equal to a Euclidean circle (but with a different centre and radius). We can see this as follows. Consider the hyperbolic circle $C_H(u,r)$. Let L be the line in $\mathbb{H}^2$ such that $F_L(u) = 0$. Since F_L is an isometry, the image of $C_H(u,r)$ under F_L is equal to $C_H(0,r)$. By Example 4.11 the hyperbolic circle $C_H(0,r)$ is equal to the Euclidean circle $x^2+y^2 = a^2$ where $r = \ln \frac{1+a}{1-a}$. The original circle $C_H(u,r)$ is equal to the image under F_L of the Euclidean circle $x^2+y^2 = a^2$, which is also a Euclidean circle by Theorem 4.6.

4.29 Theorem: *Let $u \in \mathbb{H}^2$ and let $r > 0$. The circumference L of the circle $C_H(u,r)$ and the area A of the disc $D_H(u,r)$ are given by*

$$L = 2\pi \sinh r,$$
$$A = 2\pi\big(\cosh r - 1\big).$$

Proof: By the above note, the required circumference and area are the same as the hyperbolic circumference and area of the circle $x^2+y^2 = a^2$ with $r = d_H((0,0),(a,0)) = \ln \frac{1+a}{1-a}$. Note that $\cosh r = \frac{1}{2}(e^r + e^{-r}) = \frac{1}{2}\big(\frac{1+a}{1-a} + \frac{1-a}{1+a}\big) = 1 + \frac{2a^2}{1-a^2}$, and by Example 4.12, the area of $D_H(u,r)$ is

$$A = \frac{4\pi a^2}{1-a^2} = 2\pi(\cosh r - 1).$$

From $r = \ln \frac{1+a}{1-a}$ we have $e^r = \frac{1+a}{1-a} \Longrightarrow e^r - ae^r = 1 + a \Longrightarrow e^r - 1 = a(e^r+1) \Longrightarrow a = \frac{e^r-1}{e^r-1}$. The hyperbolic circumference L of $C_H(u,r)$ is equal to the Euclidean circumference of the circle $x^2+y^2 = a^2$ scaled by the factor $\frac{2}{1-a^2}$ to give

$$L = \frac{4\pi\, a}{1-a^2} = \frac{4\pi\big(\frac{e^r-1}{e^r+1}\big)}{1-\big(\frac{e^r-1}{e^r+1}\big)^2} = \frac{4\pi(e^r-1)(e^r+1)}{(e^r+1)^2-(e^r-1)^2}$$
$$= \frac{4\pi(e^{2r}-1)}{4e^r} = 2\pi \cdot \frac{e^r - e^{-r}}{2} = 2\pi \sinh r.$$

4.30 Definition: As mentioned above, a Euclidean circle which is contained in $\mathbb{H}^2$ is also a hyperbolic circle (but with a different centre and radius). When a Euclidean circle E is contained in $\mathbb{H}^2 \cup \mathbb{S}^1$ and is tangent at one point in $\mathbb{S}^1$, the intersection $C = E \cap \mathbb{H}^2$ is called a **horocycle** in $\mathbb{H}^2$. When a Euclidean circle E intersects $\mathbb{S}^1$ at two distinct points, the intersection $C = E \cap \mathbb{H}^2$ is called a **hypercycle** in $\mathbb{H}^2$.

Angles and Triangles

4.31 Definition: Angles between curves in $\mathbb{H}^2$, and oriented angles from one directed curve in $\mathbb{H}^2$ to another, are the same as the corresponding Euclidean angles in $\mathbb{R}^2$. For example, given two smooth parametric curves $x = \alpha(t)$ and $y = \beta(t)$ in $\mathbb{H}^2$ with say $\alpha(0) = \beta(0) = p \in \mathbb{H}^2$, the **oriented angle** at p from the curve $x = \alpha(t)$ to the curve $x = \beta(t)$ is equal to $\theta_o\big(\alpha'(0), \beta'(0)\big) = \theta_o\big(\beta'(0)\big) - \theta_o\big(\alpha'(0)\big) \in [0, 2\pi)$, as in Definition 1.29, and it is determined by

$$\cos\theta_o\big(\alpha'(0), \beta'(0)\big) = \frac{\alpha'(0) \cdot \beta'(0)}{|\alpha'(0)|\,|\beta'(0)|} \quad \text{and} \quad \sin\theta_o\big(\alpha'(0), \beta'(0)\big) = \frac{\det\big(\alpha'(0), \beta'(0)\big)}{|\alpha'(0)|\,|\beta'(0)|},$$

as in Theorem 1.30, and the **unoriented angle** at p between the curves $x = \alpha(t)$ and $x = \beta(t)$ is given by

$$\theta\big(\alpha'(t), \beta'(0)\big) = \cos^{-1}\frac{\alpha'(0) \cdot \beta'(0)}{|\alpha'(0)|\,|\beta'(0)|} \in [0, \pi].$$

Given $u, v, w \in \mathbb{H}^2$ (or more generally, given $u, v, w \in \mathbb{H}^2 \cup \mathbb{S}^1$) with $u \neq v$ and $u \neq w$, we define the oriented and unoriented hyperbolic angles $\angle_o vuw$ and $\angle vuw$ as follows: let u_v be the unit tangent vector (or any tangent vector) at u to the arc along the hyperbolic line from u to v, and let u_w be the unit tangent vector (or any tangent vector) at u to the arc along the hyperbolic line from u to w, and define

$$\angle_o vuw = \theta_o(u_v, u_w) \quad \text{and} \quad \angle vuw = \theta(u_v, u_w).$$

For $u, v, w \in \mathbb{H}^2$ (or, more generally, for $u, v, w \in \mathbb{H}^2 \cup \mathbb{S}^1$) we say that u, v and w are **noncolinear** when there is no hyperbolic line which contains (or is asymptotic) to all three points. A (non-degenerate, hyperbolic) **triangle** in $\mathbb{H}^2$ (or in $\mathbb{H}^2 \cup \mathbb{S}^1$) is determined by three noncolinear points $u, v, w \in \mathbb{H}^2$ (or $u, v, w \in \mathbb{H}^2 \cup \mathbb{S}^1$) which we call the **vertices** of the triangle. When one or more of the vertices of a hyperbolic triangle lies in $\mathbb{S}^1$, the triangle is called an **asymptotic triangle** (we say it is **doubly asymptotic** when two of its vertices lie in $\mathbb{S}^1$ and **triply asymptotic** when all three vertices lie in $\mathbb{S}^1$). As with Euclidean or spherical triangles, we could think of a hyperbolic triangle in several ways: we could think of the triangle as being equal to its set of vertices $\{u, v, w\}$, or we can keep track of the order of the points and think of the triangle as an ordered triple (u, v, w), or we could think of the triangle as being the union of its three hyperbolic **edges** $[v, w]$, $[w, u]$ and $[u, v]$, (where for example $[u, v]$ denotes the arc along the hyperbolic line from u to v), or we can think of the hyperbolic triangle as the region $[u, v, w] \subseteq \mathbb{H}^2 \cup \mathbb{S}^1$ which is bounded by the three edges. We shall agree that an **ordered triangle** in $\mathbb{H}^2$ (or in $\mathbb{H}^2 \cup \mathbb{S}^1$) consists of an ordered triple (u, v, w) of noncolinear points in $\mathbb{H}^2$ (or in $\mathbb{H}^2 \cup \mathbb{S}^1$) together with the region $[u, v, w]$ which is bounded by the three edges $[v, w]$, $[w, u]$ and $[u, v]$. For this triangle, we shall normally denote the hyperbolic **edge lengths** by a, b and c with

$$a = d_H(v, w)\ ,\ b = d_H(w, u)\ ,\ c = d_H(u, v)$$

and we shall normally denote the oriented and unoriented **angles** at the vertices by α_o, β_o and γ_o and α, β and γ with

$$\alpha_o = \angle_o vuw\ ,\ \beta_o = \angle_o wvu\ ,\ \gamma_o = \angle_o uwv\ ,\ \alpha = \angle vuw\ ,\ \beta = \angle wvu\ ,\ \gamma = \angle uwv.$$

The unoriented angles α, β and γ are also called the **interior angles** of the triangle, and the **exterior angles** are given by $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$.

4.32 Example: Let $u=\left(\frac{1}{2},0\right)$, $v=\left(\frac{1}{2},\frac{1}{2}\right)$ and $w=\left(-\frac{1}{2},-\frac{1}{2}\right)$. In the hyperbolic triangle $[u,v,w]$, find the edge length $b=d_H(w,u)$ and the oriented angle $\beta_o=\angle_o wvu$.

Solution: The edge length b is given by

$$b=d_H(w,u)=\cosh^{-1}\left(1+\frac{2|w-u|^2}{(1-|w|^2)(1-|u|^2)}\right)=\cosh^{-1}\left(1+\frac{2\cdot\frac{5}{4}}{\frac{1}{2}\cdot\frac{3}{4}}\right)=\cosh^{-1}\frac{23}{3}\,.$$

To find the oriented angle β_o at the vertex v, we shall find v_u and v_w. The hyperbolic line L through v and w is the same as the Euclidean line L through v and w (since it passes through the origin), namely the line $y=x$. Since L has slope 1, we have $v_w=(-1,-1)$ (or some positive multiple of that). Let N be the hyperbolic line through u and w, say $N=C_E(p,r)\cap\mathbb{H}^2$ with $p=(x,y)$. To have $u\in N$, as in the proof of Theorem 4.18 we need $p\cdot u=\frac{|u|^2+1}{2}$, that is $\frac{1}{2}x=\frac{\frac{1}{4}+1}{2}=\frac{5}{8}$ (1). To have $v\in N$ we need $p\cdot v=\frac{|v|^2+1}{2}$, that is $\frac{1}{2}x=\frac{1}{2}y=\frac{\frac{1}{2}+1}{2}=\frac{3}{4}$, or equivalently $x+y=\frac{3}{2}$ (2). Solve Equations (1) and (2) to get $p=(x,y)=\left(\frac{5}{4},\frac{1}{4}\right)$. We also remark (even though we do not need it for our calculations) that as in Note 4.14, we must have $r=\sqrt{|p|^2-1}=\frac{\sqrt{10}}{4}$. Since the radius $v-p=\left(-\frac{3}{4},\frac{1}{4}\right)$ has slope $-\frac{1}{3}$, the tangent to N at v has slope 3, so we have $v_u=(-1,-3)$ (or any positive multiple of that). Thus $\beta_o=\theta_o(v_w,v_u)=\theta_o\big((-1,-1),(-1,-3)\big)$. Since $\cos\beta_o=\frac{v_w\cdot v_u}{|v_w|\,|v_u|}=\frac{(-1,-1)\cdot(-1,-3)}{|(-1,-1)|\,|(-1,-1)|}=\frac{4}{\sqrt{2}\sqrt{10}}=\frac{2}{\sqrt{5}}$, and $\det(v_w,v_u)=\det\begin{pmatrix}-1&-1\\-1&-3\end{pmatrix}=2$ so that $\sin\beta_o>0$, we have $\beta_o=\cos^{-1}\frac{2}{\sqrt{5}}=\sin^{-1}\frac{1}{\sqrt{5}}=\tan^{-1}\frac{1}{2}$.

4.33 Theorem: *For a triangle in $\mathbb{H}^2$ with side lengths a, b and c and interior angles α, β and γ, we have*

(1) (The Sine Law) $$\frac{\sinh a}{\sin\alpha}=\frac{\sinh b}{\sin\beta}=\frac{\sinh c}{\sin\gamma},$$

(2) (The First Cosine Law) $$\cos\alpha=\frac{\cosh a-\cosh b\cosh c}{-\sinh b\sinh c}\ ,\ \textit{and}$$

(3) (The Second Cosine Law) $$\cosh a=\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

Similar rules hold with a,b,c and α,β,γ permuted.

Proof: Use a hyperbolic reflection to move the vertex u to the position $(0,0)$, then use another reflection to move the vertex v to position $(s,0)$ with $0<s<1$, and use a third reflection to move the vertex w to the upper half of $\mathbb{H}^2$ at position $(t\cos\alpha,t\sin\alpha)$ with $0<t<1$. Note that the reflections preserve the edge lengths and angles, so it suffices to prove that the Sine Law and Cosine Laws hold for a triangle with vertices at $u=(0,0)$, $v=(s,0)$ and $w=(t\cos\alpha,t\sin\alpha)$. For this triangle, α is the angle at u and we have

$$\cosh b=\cosh\big(d_H(w,u)\big)=1+\frac{2|w-u|^2}{(1-|w|^2)(1-|u|^2)}=1+\frac{2t^2}{1-t^2}=\frac{1+t^2}{1-t^2}$$
$$\sinh b=\sqrt{\cosh^2 b-1}=\sqrt{\frac{1+2t^2+t^4}{1-2t^2+t^4}-1}=\sqrt{\frac{4t^2}{1-2t^2+t^4}}=\frac{2t}{1-t^2}$$

and similarly $\cosh c=\frac{1+s^2}{1-s^2}$ and $\sinh c=\frac{2s}{1-s^2}$. Also, by the Euclidean Law of Cosines, we have $|v-w|^2=s^2+t^2-2st\cos\alpha$ so that

$$\cosh a=\cosh\big(d_H(v,w)\big)=1+\frac{2|v-w|^2}{(1-|v|^2)(1-|w|^2)}=1+\frac{2(s^2+t^2-2st\cos\alpha)}{(1-s^2)(1-t^2)}$$
$$=\frac{(1-s^2)(1-t^2)+2(s^2+t^2)-4st\cos\alpha}{(1-s^2)(1-t^2)}=\frac{(1+s^2)(1+t^2)-4st\cos\alpha}{(1-s^2)(1-t^2)}.$$

Thus we have

$$\frac{\cosh b\cosh c-\cosh a}{\sinh b\sinh c}=\frac{\left(\frac{1+t^2}{1-t^2}\right)\left(\frac{1+s^2}{1-s^2}\right)-\frac{(1+s^2)(1+t^2)-4st\cos\alpha}{(1-s^2)(1-t^2)}}{\left(\frac{2t}{1-t^2}\right)\left(\frac{2s}{1-s^2}\right)}=\cos\alpha$$

proving the First Cosine Law.

To prove the Sine Law, let us find $\sin\beta$ and $\sin\gamma$. Let L be the hyperbolic line through v and w, say $L=C_E(p,r)$ with $p=(x,y)$ and $r=\sqrt{|p|^2-1}$. To get $v\in L$ we need $p\cdot v=\frac{|v|^2+1}{2}$, that is $sx=\frac{s^2+1}{2}$, so we have $x=\frac{1+s^2}{2s}$. To have $w\in L$ we need $p\cdot w=\frac{|w|^2+1}{2}$, that is $(t\cos\alpha)x+(t\sin\alpha)y=\frac{t^2+1}{2}$ so we have $y=\frac{1}{\sin\alpha}\left(\frac{1+t^2}{2t}-x\cos\alpha\right)=\frac{1}{\sin\alpha}\left(\frac{1+t^2}{2t}-\frac{1+s^2}{2s}\cos\alpha\right)$. Thus we have

$$p=(x,y)=\left(\frac{1+s^2}{2s}\,,\,\frac{1}{\sin\alpha}\left(\frac{1+t^2}{2t}-\frac{1+s^2}{2s}\cos\alpha\right)\right)\ \text{ and }\ r=\sqrt{|p|^2-1}.$$

Note also that $x-s=\frac{1+s^2}{2s}-s=\frac{1-s^2}{2s}>0$. The radius vector from v to p is $p-v=(x-s,y)$ and so we have $v_w=(-y,x-s)$ with $|v_w|=|p-v|=r$. We also have $v_u=(-1,0)$ with $|v_u|=1$, so that

$$\cos\beta=\frac{v_w\cdot v_u}{|v_w|\,|v_u|}=\frac{y}{r}.$$

Also note that $r^2=|p-v|^2=(x-s)^2+y^2$, so we have

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\frac{y^2}{r^2}}=\sqrt{\frac{r^2-y^2}{r^2}}=\sqrt{\frac{(x-s)^2}{r^2}}=\frac{x-s}{r}=\frac{1-s^2}{2rs}.$$

By symmetry, that is by interchanging the roles of v and w, we also have $\sin\gamma=\frac{1-t^2}{2rt}$ so that

$$\frac{\sinh b}{\sin\beta}=\frac{\frac{2t}{1-t^2}}{\frac{1-s^2}{2rs}}=\frac{2rst}{(1-s^2)(1-t^2)}=\frac{\frac{2s}{1-s^2}}{\frac{1-t^2}{2rt}}=\frac{\sinh c}{\sin\gamma}$$

proving the Sine Law.

For the Second Cosine Law, let us also find $\cos\gamma$. The radius vector from p to w is $w-p=\big(t\cos\alpha-x\,,\,t\sin\alpha-y\big)$ so we have $w_v=\big(y-t\sin\alpha,t\cos\alpha-x\big)$ with $|w_v|=|w-p|=r$, and we have $w_u=(-\cos\alpha,-\sin\alpha)$ with $|w_u|=1$, and hence

$$\cos\gamma=\frac{w_u\cdot w_v}{|w_u|\,|w_v|}=\frac{x\sin\alpha-y\cos\alpha}{r}.$$

Thus, making use of many of the above formulas, including the formulas $(x-s)^2+y^2=r^2$ and $y=\frac{1}{\sin\alpha}\left(\frac{1+t^2}{2t}-x\cos\alpha\right)$ and $x=\frac{1+s^2}{2s}$ so that $2sx-s^2=1$, we have

$$\begin{aligned}\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}&=\frac{\cos\alpha+\frac{y}{r}\cdot\frac{x\sin\alpha-y\cos\alpha}{r}}{\frac{1-s^2}{2rs}\cdot\frac{1-t^2}{2rt}}=\frac{4r^2st\cos\alpha+4st\,xy\sin\alpha-4sty^2\cos\alpha}{(1-s^2)(1-t^2)}\\&=\frac{4st\Big((x-s)^2\cos\alpha+x\Big(\frac{1+t^2}{2t}-x\cos\alpha\Big)\Big)}{(1-s^2)(1-t^2)}=\frac{4st\Big(x\,\frac{1+t^2}{2t}-(2sx-s^2)\cos\alpha\Big)}{(1-s^2)(1-t^2)}\\&=\frac{4st\Big(\Big(\frac{1+s^2}{2s}\Big)\Big(\frac{1+t^2}{2t}\Big)-\cos\alpha\Big)}{(1-s^2)(1-t^2)}=\frac{(1+s^2)(1+t^2)-4st\cos\alpha}{(1-s^2)(1-t^2)}=\cosh a\end{aligned}$$

proving the Second Cosine Law.

4.34 Exercise: Let $u=\left(\frac{1}{2},0\right)$, $v=\left(\frac{1}{2},\frac{1}{2}\right)$ and $w=\left(-\frac{1}{2},-\frac{1}{2}\right)$ (as in Example 4.32). In the hyperbolic triangle $[u,v,w]$, find a, b and c, then find $\cos\beta$ using the First Cosine Law.

4.35 Lemma: *The area of a doubly asymptotic triangle in* $\mathbb{H}^2\cup\mathbb{S}^1$ *with interior angle* α *at its non-asymptotic vertex is equal to* $A=\pi-\alpha$.

Proof: Consider a doubly asymptotic triangle with angle $\alpha=2\beta$ at its non-asymptotic vertex. We can use hyperbolic reflections to move the triangle so that the non-asymptotic vertex is at the origin and the asymptotic vertices are $u=(\cos\beta,\sin\beta)$ and $v=(\cos\beta,-\sin\beta)$. Note that the hyperbolic line L which is asymptotic to u and v is equal to $L=C\cap\mathbb{H}^2$ where C is the circle of radius $r=\tan\beta$ centred at $a=(\sec\beta,0)$. For a point $x=(r\cos\theta,r\sin\theta)\in L$ with $-\beta\le\theta\le\beta$ and $0\le r\le 1$, the Law of Cosines applied to the triangle with vertices at $0,a,x$ and the Quadratic Formula give

$$\begin{gathered}\tan^2\beta=r^2+\sec^2\beta-2r\sec\beta\,\cos\theta\\ r^2-2r\sec\beta\cos\theta+1=0\\ r=\sec\beta\cos\theta\pm\sqrt{\sec^2\beta\cos^2\theta-1}.\end{gathered}$$

For $-\beta\le\theta\le\beta$ we have $\cos\theta\ge\cos\beta$ so that $\sec\beta\cos\theta\ge 1$, and so in order to have $r\le 1$ we must use the negative sign. Thus the line L is given in polar coordinates by

$$r=\sec\beta\cos\theta-\sqrt{\sec^2\beta\cos^2\theta-1}.$$

Thus the area of the doubly asymptotic triangle is

$$\begin{aligned}
A&=\int_{\theta=-\beta}^{\beta}\int_{r=0}^{\sec\beta\cos\theta-\sqrt{\sec^2\beta\cos^2\theta-1}}\frac{4r}{(1-r^2)^2}\;dr\,d\theta\\
&=\int_{\theta=-\beta}^{\beta}\left[\frac{2}{1-r^2}\right]_{r=0}^{\sec\beta\cos\theta-\sqrt{\sec^2\beta\cos^2\theta-1}}d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{2}{1-\big(\sec\beta\cos\theta-\sqrt{\sec^2\beta\cos^2\theta-1}\big)^2}-2\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{2}{1-\big(2\sec^2\beta\cos^2\theta-1-2\sec\beta\cos\theta\sqrt{\sec^2\beta\cos^2\theta-1}\big)}-2\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{1}{-\big(\sec^2\beta\cos^2\theta-1\big)+\sec\beta\cos\theta\sqrt{\sec^2\beta\cos^2\theta-1}}-2\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{-(\sec^2\beta\cos^2\theta-1)-\sec\beta\cos\theta\sqrt{\sec^2\beta\cos^2\theta-1}}{\big(\sec^2\beta\cos^2\theta-1\big)^2-\sec^2\beta\cos^2\theta\big(\sec^2\beta\cos^2\theta-1\big)}-2\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{-\big(\sec^2\beta\cos^2\theta-1\big)-\sec\beta\cos\theta\sqrt{\sec^2\beta\cos^2\theta-1}}{\big(\sec^2\beta\cos^2\theta-1\big)\big(\sec^2\beta\cos^2\theta-1-\sec^2\beta\cos^2\theta\big)}-2\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}1+\frac{\sec\beta\cos\theta}{\sqrt{\sec^2\beta\cos^2\theta-1}}-2\;d\theta=\int_{\theta=-\beta}^{\beta}\frac{\cos\theta}{\sqrt{\cos^2\theta-\cos^2\beta}}-1\;d\theta\\
&=\int_{\theta=-\beta}^{\beta}\frac{\cos\theta}{\sqrt{\sin^2\beta-\sin^2\theta}}\;d\theta\;-2\beta=\int_{\phi=-\pi/2}^{\pi/2}\frac{\sin\beta\cos\phi\;d\phi}{\sin\beta\cos\phi}\;-\alpha=\pi-\alpha
\end{aligned}$$

where on the last line we made the trigonometric substitution $\sin\beta\sin\phi=\sin\theta$ so that $\sqrt{\sin^2\beta-\sin^2\theta}=\sin\beta\cos\phi$ and $\cos\theta\,d\theta=\sin\beta\cos\phi\,d\phi$.

4.36 Theorem: *The area of a triangle in* $\mathbb{H}^2$ *(or* $\mathbb{H}^2 \cup \mathbb{S}^1$*) with interior angles* α*,* β *and* γ *is equal to*

$$A = \pi - (\alpha + \beta + \gamma).$$

This includes asymptotic triangles with one or more vertices on $\mathbb{S}^1$ *(the interior angles at asymptotic points are equal to zero).*

Proof: This follows from the above lemma because a triply asymptotic triangle can be cut into two doubly asymptotic triangles, and given a singly asymptotic triangle we can add a doubly asymptotic triangle to form a doubly asymptotic triangle, and given a non-asymptotic triangle, we can add three doubly asymptotic triangles to form a triply asymptotic triangle.

Isometries

4.37 Definition: Let L and M be two distinct lines in $\mathbb{H}^2$. When L and M intersect at a point $p \in \mathbb{H}^2$ and $\theta = 2\varphi$ where φ is the oriented angle from L counterclockwise to M at p, the isometry $R_{p,\theta} = F_M F_L$ is called the **rotation** about p by θ in $\mathbb{H}^2$. When L and M are asyptotic at the point $p \in \mathbb{S}^1$, the isometry $P = F_M F_L$ is called a **horolation** (or a **parallel displacement**) about p in $\mathbb{H}^2$. When L and M do not intersect and are not asymptotic, and N is the unique line which intersects orthogonally with L and M, the isometry $T = F_M F_L$ is called a **translation** along N in $\mathbb{H}^2$, and the isometry $F_N F_M F_L$ is called a **glide reflection** along N in $\mathbb{H}^2$.

4.38 Remark: A rotation on $\mathbb{H}^2$ is also called an **elliptic isometry** on $\mathbb{H}^2$, a horolation on $\mathbb{H}^2$ is also called a **parabolic isometry** on $\mathbb{H}^2$, and a translation on $\mathbb{H}^2$ is also called a **hyperbolic isometry** on $\mathbb{H}^2$.

4.39 Remark: A rotation about the point $p \in \mathbb{H}^2$ moves each point along a (hyperbolic) circle centred at p. A horolation about the point $p \in \mathbb{S}^1$ moves each point along a horocycle at p (that is a Euclidean circle in $\mathbb{H}^2$ which is tangent to $\mathbb{S}^1$ at p). A translation along the line L in $\mathbb{H}^2$ which is asymptotic to $u, v \in \mathbb{S}^1$, moves each point along a hypercycle from u to v (that is along an arc of a Euclidean circle through u and v).

4.40 Theorem: *Let $u, v \in \mathbb{H}^2$ with $u \neq v$. Let L be the perpendicular bisector of u and v in $\mathbb{H}^2$. Then for $x \in \mathbb{H}^2$ we have $d_H(x,u) = d_H(x,v) \iff x \in L$.*

Proof: If $x \in L$ then $F_L(x) = x$ (from Definition 4.1 or from Part 1 of Theorem 4.6) and so, since F_L is an isometry, we have $d_H(x,u) = d_H\big(F_L(x), F_L(u)\big) = d_H(x,v)$.

Recall that (in the statement of Theorem 4.20) we defined the perpendicular bisector L to be the hyperbolic line such that $F_L(u) = v$. Let M be the hyperbolic line through u and v and let m be the point of intersection of L with M. Note that since $m \in L$, we have $d_H(m,u) = d_H(m,v)$ (as shown above), and so m is the hyperbolic midpoint of the hyperbolic line segment $[u,v]$. Since $F_L(u) = v$ and $F_L(m) = m$, we have $F_L(M) = M$ and F_L sends the hyperbolic line segment $[u,m]$ to the hyperbolic line segment $[v,m]$ with $\angle umv = \pi$. Since F_L preserves angles, the angle between L and M at m is equal to $\frac{\pi}{2}$ (for $p \in L$ with $p \neq m$ we have $\angle ump = \angle vmp$ and $\angle ump + \angle vmp = \pi$). This shows that the perpendicular bisector L can also be described as the line through the hyperbolic midpoint m of $[u,v]$ which is orthogonal to $[u,v]$.

Let $x \in \mathbb{H}^2$ with $d_H(x,u) = d_H(x,v)$. Note that in the two hyperbolic triangles $[x,u,m]$ and $[x,v,m]$, the corresponding edge lengths are all equal and hence, by the First Law of Cosines, the corresponding interior angles are all equal. In particular, we have $\angle umx = \angle vmx$. Since m lies between u and v, we also have $\angle umx + \angle vmx = \pi$, and so $\angle umx = \angle vmx = \frac{\pi}{2}$. Thus x lies on the line through m which is orthogonal to $[u,v]$, so $x \in L$, as required.

4.41 Theorem: *Let $[u,v,w]$ be a triangle in $\mathbb{H}^2$ (so the points $u, v, w \in \mathbb{H}^2$ are non-colinear). Then a point $x \in \mathbb{H}^2$ is uniquely determined by the distances $d_H(x,u)$, $d_H(x,v)$ and $d_H(x,w)$.*

Proof: Let $x, y \in \mathbb{H}^2$ with $x \neq y$ and suppose, for a contradiction, that $d_H(x,u) = d_H(y,u)$ and $d_H(x,v) = d_H(y,v)$ and $d_H(x,w) = d_H(y,w)$. Let L be the perpendicular bisector of $[x,y]$ in $\mathbb{H}$. By the above theorem, since $d_H(x,u) = d_H(y,u)$ we have $u \in L$, and since $d_H(x,v) = d_H(y,v)$ we have $v \in L$, and since $d_H(x,w) = d_H(y,w)$ we have $w \in L$, which contradicts the fact that u, v and w are non-colinear.

4.42 Theorem: *Let $[u, v, w]$ and $[u', v', w']$ be ordered triangles in $\mathbb{H}^2$ with corresponding edge lengths equal, that is with $a = a'$, $b = b'$ and $c = c'$. Then there exists a unique isometry F on $\mathbb{H}^2$ such that $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$.*

Proof: The uniqueness of such an isometry follows from the previous theorem. Indeed assuming that such an isometry F exists, then given any point $x \in \mathbb{H}^2$, the point $y = F(x)$ is the unique point $y \in \mathbb{H}^2$ such that $d_H(y, u') = d_H(x, u)$, $d_H(y, v') = d_H(x, v)$ and $d_H(y, w') = d_H(x, w)$.

It remains to show that such an isometry on $\mathbb{H}^2$ exists. If $u = u'$ then let F_1 be the identity map, and if $u \neq u'$ then let F_1 be the hyperbolic reflection in the perpendicular bisector L of u and u'. Let $u_1 = F_1(u) = u'$, $v_1 = F_1(v)$ and $w_1 = F_1(w)$. If $v_1 = v'$ then let F_2 be the identity map, and if $v_1 \neq v'$ then let F_2 be the hyperbolic reflection in the perpendicular bisector M of v_1 and v'. Note that since $d_H(u_1, v_1) = d_H(u, v) = d_H(u', v') = d_H(u_1, v')$ we have $u_1 \in M$ so that $F_M(u_1) = u_1 = u'$. Let $u_2 = F_2(u_1) = u'$, $v_2 = F_2(v_1) = v'$ and $w_2 = F_2(w_1)$. If $w_1 = w'$ then let F_3 be the identity map, and if $w_2 \neq w'$ then let F_3 be the hyperbolic reflection in the perpendicular bisector N of w_2 and w'. As above, since $d_H(u_2, w_2) = d_H(u, w) = d_H(u', w') = d_H(u_2, w')$ we have $u_2 \in N$ so that $F_N(u_2) = u_2 = u'$, and since $d_H(v_2, w_2) = d_H(v, w) = d_H(v', w') = d_H(v_2, w')$ we have $v_2 \in N$ so that $F_N(v_2) = v_2 = v'$. Thus we can let F be the composite $F = F_3 F_2 F_1$ and then we have $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$, as required.

4.43 Theorem: *Every isometry on $\mathbb{H}^2$ is equal to a product of 0, 1, 2 or 3 reflections.*

Proof: Let F be any isometry on $\mathbb{H}^2$. Let $u = (0, 0)$, $v = \left(\frac{1}{2}, 0\right)$ and $w = \left(0, \frac{1}{2}\right)$, and let $u' = F(u)$, $v' = F(v)$ and $w' = F(w)$. The proof of the previous theorem shows that $F = F_3 F_2 F_1$ where each F_k is equal either to the identity map or to a hyperbolic reflection (the product of zero reflections is the identity map, which occurs when all three of the maps F_k is the identity map).

4.44 Theorem: *Every isometry on $\mathbb{H}^2$ is equal to the identity, a rotation, a translation, a parallel displacement, or a reflection or a glide reflection.*

Proof: Every isometry is the product of 0, 1, 2 or 3 reflections, and the product of 0 reflections is the identity map, the product of 1 reflections is a reflection, and the product of 2 reflections (by definition) is a rotation, a translation or a parallel displacement. It remains to consider the product of 3 reflections. Suppose that $F = F_N F_M F_L$. If $M = L$ then we have $F = F_N$, which is a reflection. Suppose that $M \neq L$. There are three cases to consider: either M and L intersect in $\mathbb{H}^2$, or M and L are asymptotic, or M and L are parallel. We shall consider only the first case, and leave the other two cases as an exercise.

Case 1: suppose that $L \cap M = \{a\}$ and $F_M F_L = R_{a,\theta}$. Let $N' = N$, let M' be the (unique) hyperbolic line through a which is perpendicular to $N = N'$, say M' intersects N' at b, and let L' be the (unique) hyperbolic line through a such that the oriented angle from L' to M' is equal to $\frac{\theta}{2}$ so that $R_{a,\theta} = F_{M'} F_{L'}$. Then we have $F = F_N F_M F_L = F_N R_{a,\theta} = F_{N'} F_{M'} F_{L'} = R_{b,\pi} F_{L'}$. Let $L'' = L'$, let N'' be the (unique) hyperbolic line through b perpendicular to L'', and let M'' be the (unique) hyperbolic line through b perpendicular to N'' so that $R_{b,\pi} = F_{N''} F_{M''}$. Then we have $F = R_{b,\pi} F_{L'} = F_{N''} F_{M''} F_{L''}$ where L'' and M'' are both perpendicular to N''. If $L'' = M''$ then $F = F_{N''}$ which is a reflection, and if $L'' \neq M''$ the F is a glide refection along N''.

For the other cases, first show that given a line L through $u \in \mathbb{S}^1$ and a parallel displacement P about u, there is a line M such that $P = F_M F_M$, and given orthogonal lines L and K and a translation T along K, there is a line M such that $T = F_M F_L$.

The Half-Plane Model, the Minkowski Model, and the Klein Model

We have described the Poincaré disc model of the hyperbolic plane, but there are several other models of the hyperbolic plane that are sometimes used: there are alternate ways of constructing a geometry (a set with a an abstract way of measuring distance between points) in which we can define lines and circles and triangles which have have the same properties and satisfy the same formulas as lines circles and triangles in the Poincaré disc (for example the formula for the area of a disc, the laws of cosines, and the formula for the area of a triangle). Here is a brief description of three such models.

The Poincaré **upper half plane model** of the hyperbolic plane is constructed as follows. Let $\mathbb{U}^2$ be the upper half plane $\mathbb{U}^2 = \{(x,y)\big|y>0\}$. Let C be the Euclidean circle $C = C_E\big((0,1),\sqrt{2}\big)$ and let L be the Euclidean line $y=0$ (that is the x-axis). Note that F_C sends the unit circle $\mathbb{S}^1$ (with the point $(0,1)$ removed) to the x-axis with $F_C(1,0)=(1,0)$, $F_C(-1,0)=(-1,0)$, $F_C(0,-1)=(0,0)$ and $F_C(0,1)$ undefined, and F_C sends the disc $\mathbb{H}^2$ to the lower half plane $y<0$. The composite $S=F_LF_C$ sends the disc $\mathbb{H}^2$ to the upper half plane $\mathbb{U}^2$. The inverse of S is given by $T=S^{-1}=F_CF_L$. In the upper half plane model of the hyperbolic plane, we define the distance between two points in the upper half plane in order to make the map S an isometry, so for $u,v\in\mathbb{H}^2$ we define $d_U\big(u,v\big)=d_H\big(T(u),T(v)\big)$. The maps S and T are conformal (they preserve angles between the curves), so the angles between two curves in $\mathbb{U}^2$ are equal to the Euclidean angles between the curves. The geodesics (lines which minimize distance) in $\mathbb{H}^2$ are mapped by S to the geodesics in $\mathbb{U}^2$. The geodesics in $\mathbb{H}^2$ are the straight lines through 0 and the arcs along circles which are orthogonal to $\mathbb{S}^1$, and the geodesics in $\mathbb{U}^2$ are the vertical lines and the upper half circles which intersect orthogonally with the x–axis.

The **Minkowski model**, also called the **hyperboloid model**, of the hyperbolic plane is one half of a hyperboloid in 3-dimensional **Minkowski space**. The 3-dimensional Minkowski space is the set $\mathbb{R}^3$ using a different norm. The standard Euclidean quadratic form in $\mathbb{R}^3$ (the square of the norm) is given by $N(x,y,z)=x^2+y^2+z^2$, and the Minkowski quadratic form in $\mathbb{R}^3$ is given by $Q(x,y,t)=x^2+y^2-t^2$ (which can take negative values). Let $\mathbb{M}^2$ be the upper sheet of the hyperboloid $Q(x,y,t)=-1$, that is let

$$\mathbb{M}^2=\big\{(x,y,t)\in\mathbb{R}^3\,\big|\,x^2+y^2-t^2=-1\,,\,t>0\big\}.$$

We can define a projection $S:\mathbb{M}^2\to\mathbb{H}^2$, similar to the stereographic projection, as follows: given $(x,y,t)\in\mathbb{M}^2$, we let $(u,v)\in\mathbb{H}^2$ be the point such that the line in $\mathbb{R}^3$ through $(0,0,-1)$ and (x,y,t) intersects the xy-plane at the point $(u,v,0)$. In the Minkowski model of the hyperbolic plane, we define the distance between two points in $\mathbb{M}^2$ in order to make the map S an isometry. The geodesics in $\mathbb{M}^2$ are the curves of intersection of $\mathbb{M}^2$ with a plane in $\mathbb{R}^3$ through the origin.

The **Klein model** of the hyperbolic plane is constructed as follows. Let $\mathbb{K}^2$ be the unit disc $\mathbb{K}^2=\big\{x,y)\big|\,x^2+y^2<1\big\}$ (so in fact $\mathbb{K}^2=\mathbb{H}^2$, but the distance between two points in $\mathbb{K}^2$ is not the same as the distance between the same two points in $\mathbb{H}^2$ or between the same two points in $\mathbb{R}^2$). Define another projection $S:\mathbb{M}^2\to\mathbb{K}^2$, similar to the stereographic projection, as follows: given $(x,y,t)\in\mathbb{M}^2$ we let $(u,v)\in\mathbb{K}^2$ be the point such that the line in $\mathbb{R}^3$ through $(0,0,0)$ and (x,y,t) intersects the plane $z=1$ at the point $(u,v,1)$. We define distance in $\mathbb{K}^2$ so that the map S is an isometry. The geodesics in $\mathbb{K}^2$ are segments along straight lines (but the hyperbolic angle between two lines in $\mathbb{K}^2$ is not the same as the hyperbolic angle between the same two lines in $\mathbb{R}^2$).

Tilings

This topic will not be covered (but I may include notes later).