Chapter 3. Projective Geometry

The Projective Plane

3.1 Note: The rudiments of projective geometry were first studied by artists who were interested in perspective drawing. Imagine drawing a picture of a scene as follows. Set up a system of coordinates with the artist's eye at the origin, the z-axis pointing upwards, so that the artist is looking in the direction of the y-axis. Erect a plane of glass along the plane $y = 1$. All points in space which lie along the same ray through the origin will appear to be at the same position relative to the artist's eye. An object at position (x, y, z) with $y > 0$ appears to be at the same position as the point at position $\left(\frac{x}{y}, 1, \frac{z}{y}\right)$ which lies on the pane of glass so the artist draws a spot at this point on the plane of glass.

3.2 Example: An artist views a railway track which lies in the plane $z = -1$ with rails along the lines $x = \pm 1$ and ties along the line segments $-1 \leq x \leq 1$, $y = k$, $k \in \mathbb{Z}^+$. The artist sets up a pane of glass along the plane $y = 1$. The endpoints of the ties, which are at $(x, y, z) = (\pm 1, k, -1)$, will be represented by spots at position $(x, z) = (\pm\frac{1}{k}, -\frac{1}{k})$, $y = 1$. On the pane of glass, the rails are represented by the rays $z = \pm x$, $z < 0$ and the ties are represented by the line segments from $(x, z) = \left(-\frac{1}{k}, -\frac{1}{k}\right)$ to $(x, z) = \left(\frac{1}{k}, -\frac{1}{k}\right)$.

3.3 Note: From the point of view of producing a perspective drawing, all points which lie along the same ray from the origin can be identified as being the same point. Each ray through the origin intersects a unique point on the sphere, so we can identify the set of all such rays with the sphere $\mathbb{S}^2$. The rays which are visible to an artist looking in the direction of the positive y-axis are then identified with the hemisphere $H = \left\{(x, y, z) \in \mathbb{S}^2 \middle| y > 0\right\}$, Notice that the map which sends a ray, or a point in H, to the corresponding point drawn by the artist on the pane of glass at $y = 1$ is the gnomic projection $\phi(x, y, z) = \left(\frac{x}{y}, \frac{z}{y}\right)$.

In projective geometry, rather than considering the set of all *rays* through the origin, we consider instead the set of all *lines* through the origin.

3.4 Definition: The real **projective plane**, denoted by $\mathbb{P}^2$, is the set of lines through the origin in $\mathbb{R}^3$. Given $0 \neq x \in \mathbb{R}^3$ we let $[x]$ denote the line in $\mathbb{R}^3$ through 0 and x, that is $[x] = \text{Span}\{x\}$, so we have

$$\mathbb{P}^2 = \left\{[x] \middle| 0 \neq x \in \mathbb{R}^3\right\}.$$

A line through the origin in $\mathbb{R}^3$ is called a **point** in $\mathbb{P}^2$. Given two lines u and v through the origin in $\mathbb{R}^3$, we define the (projective) **distance** between the points u and v in $\mathbb{P}^2$, denoted by $d_P(u, v)$, to be the angle between the lines u and v in $\mathbb{R}^3$. When $u = [x]$ and $v = [y]$ with $0 \neq x, y \in \mathbb{R}^3$, we have

$$\begin{aligned} d_P(u, v) &= \min\left\{\theta(x, y)\,,\, \theta(x, -y)\right\} = \min\left\{\theta(x, y)\,,\, \pi - \theta(x, y)\right\} \\ &= \cos^{-1}\frac{|x \cdot y|}{|x||y|} = \sin^{-1}\frac{|x \times y|}{|x||y|}. \end{aligned}$$

3.5 Note: A line u through the origin in $\mathbb{R}^3$ intersects the sphere $\mathbb{S}^2$ in two antipodal points $\pm x$, and these points determine the line, indeed $u = [x] = [-x]$. We often identify the line u with the pair of antipodal points $\pm x$ and consider $\mathbb{P}^2$ to be the set of all pairs of antipodal points in $\mathbb{S}^2$, that is

$$\mathbb{P}^2 = \left\{\{\pm x\} \middle| x \in \mathbb{S}^2\right\}.$$

3.6 Theorem: *(Metric Properties of Distance) Let $u, v, w \in \mathbb{P}^2$. Then*

(1) (Positive Definiteness) $d_P(u,v) \in \left[0, \frac{\pi}{2}\right]$ with $d_P(u,v) = 0$ if and only if $u = v$,
(2) (Symmetry) $d(u,v) = d_P(v,u)$, and
(3) (Triangle Inequality) $d_P(u,v) + d_P(v,w) \geq d_P(u,w)$.

Proof: We prove Part (3). Choose $x, y, z \in \mathbb{S}^2$ so that $u = [x]$, $v = [y]$ and $w = [z]$ and also (by replacing x and z by $\pm x$ and $\pm z$ if necessary) such that $x \cdot y \geq 0$ and $y \cdot z \geq 0$. Then

$$\begin{aligned}
\cos\Big(d_P(u,v), d_P(v,w)\Big) &= \cos d_P(u,v) \cos d_P(v,w) - \sin d_P(u,v) \sin d_P(v,w) \\
&= (x \cdot y)(y \cdot z) - |x \times y||y \times z| \text{ , since } x \cdot y \geq 0 \text{ and } y \cdot z \geq 0, \\
&\leq (x \cdot y)(y \cdot z) - \big|(x \times y) \cdot (y \times z)\big| \text{ , by the Cauchy-Schwearz Inequality} \\
&\leq (x \cdot y)(y \cdot z) - (x \times y) \cdot (y \times z) \\
&= (x \cdot y)(y \cdot z) - (x \cdot y)(y \cdot z) + (x \cdot z) \\
&= (x \cdot z) \leq |x \cdot z| = \cos d_P(u,w)
\end{aligned}$$

3.7 Definition: For $u \in \mathbb{P}^2$ and $r \in \left[0, \frac{\pi}{2}\right]$, the (projective) **circle** centred at u of radius r and the (closed projective) disc centred at u of raduis r are the sets

$$\begin{aligned}
C(u,r) &= \left\{ v \in \mathbb{P}^2 \middle| d_P(u,v) = r \right\} \text{ and} \\
D(u,r) &= \left\{ v \in \mathbb{P}^2 \middle| d_P(u,v) \leq r \right\}.
\end{aligned}$$

3.8 Note: Let $a \in \mathbb{S}^2$ and let $u = [a]$. The union of the lines $v \in C(u,r) \subseteq \mathbb{P}^2$ forms a double cone in $\mathbb{R}^3$ with vertex at the origin, and this double cone intersects $\mathbb{S}^2$ in the pair of antipodal spherical circles $C(a,r)$ and $C(-a,r) = C(a, \pi - r)$. The circumference of the projective circle $C(u,r) \subseteq \mathbb{P}^2$ is equal to that of the spherical circle $C(a,r) \subseteq \mathbb{S}^2$, and the area of the projective disc $D(u,r) \subseteq \mathbb{P}^2$ is equal to that of the spherical disc $D(a,r) \subseteq \mathbb{S}^2$.

3.9 Definition: A (projective) **line** in $\mathbb{P}^2$ is the set of all lines through the origin in $\mathbb{R}^3$ which lie in some given plane through the origin in $\mathbb{R}^3$. Note that a projective line $L \subseteq \mathbb{P}^2$ determines and is determined by a Euclidean plane $P \subseteq \mathbb{R}^3$ through the origin; given a plane P, the corresponding line L is given by $L = \left\{ u \in \mathbb{P}^2 \middle| u \subseteq P \right\}$, and given a line L, the corresponding plane P is given by $P = \bigcup\limits_{u \in L} u$. Given a projective line $L \subseteq \mathbb{P}^2$ and its corresponding Euclidean plane $P \subseteq \mathbb{R}^3$, the **pole** of L is the point $u \in \mathbb{P}^2$ which, as a line through 0 in $\mathbb{R}^3$, is perpendicular to the plane P. Given a point $u \in \mathbb{P}^2$, we write L_u to denote the projective line with pole u. We remark that a projective line is the same thing as a projective circle of radius $\frac{\pi}{2}$, indeed for $u \in \mathbb{P}^2$ we have $L_u = C\left(u, \frac{\pi}{2}\right)$.

3.10 Theorem: *(Properties of Projective Lines)*

(1) Given two distinct points $u, v \in \mathbb{P}^2$, there is a unique line $L \subseteq \mathbb{P}^2$ containing u and v.
(2) Given two distinct lines $L, M \subseteq \mathbb{P}^2$ there is a unique point $u \in \mathbb{P}^2$ with $u \in L \cap M$.
(3) Given a point $u \in \mathbb{P}^2$ and a line $L \subseteq \mathbb{P}^2$ with $L \neq L_u$, there is a unique line in $\mathbb{P}^2$ which passes through u and is perpendicular to L.
(4) Given two distinct lines $L, M \subseteq \mathbb{P}^2$ there exists a unique line in $\mathbb{P}^2$ which is perpendicular to L and M.

Proof: All parts of this theorem follow immediately from properties of lines and planes through the origin in $\mathbb{R}^3$.

3.11 Definition: A (projective) **triangle** is determined by a spherical triangle. Three non-colinear points $u, v, w \in \mathbb{S}^2$ determine an ordered spherical triangle $[u, v, w] \subseteq \mathbb{S}^2$. The corresponding solid projective triangle consists of all lines through 0 in $\mathbb{R}^3$ which pass through the solid spherical triangle $[u, v, w] \subseteq \mathbb{S}^2$. Note that the union of all the lines in the projective triangle forms a double cone, with triangular cross-section, which passes through the solid triangle $[u, v, w]$ and also the antipodal triangle $[-u, -v, -w]$. When $[u, v, w]$ is positively oriented, the area, angles, and side lengths of the projective triangle are the same as those of the spherical triangle $[u, v, w]$.

3.12 Note: We do not consider a projective triangle to have an orientation because an ordered spherical triangle $[u, v, w]$ and its antipodal triangle $[-u, -v, -w]$ each determine the same projective triangle, but these two spherical triangles have the opposite orientation.

3.13 Note: For a spherical triangle $[u, v, w]$, the edge lengths are the same as the distances between the vertices, for example $a = d_S(v, w)$, but this is not necessarily the case for the corresponding projective triangle. Indeed when $a = d_S(v, w) > \frac{\pi}{2}$ we find that $d_P\big([v], [w]\big) = d_S(v, -w) = \pi - d_S(v, w) = \pi - a$.

3.14 Definition: An **isometry** on $\mathbb{P}^2$ is a bijective map $F : \mathbb{P}^2 \to \mathbb{P}^2$ which preserves distance, that is such that for all $u, v \in \mathbb{P}^2$ we have $d_P\big(F(u), F(v)\big) = d_P(u, v)$. Note that every isometry on $\mathbb{S}^2$ determines an isometry on $\mathbb{P}^2$ as follows. Given an isometry $F : \mathbb{S}^2 \to \mathbb{S}^2$, extend F to the orthogonal map $F : \mathbb{R}^3 \to \mathbb{R}^3$ and note that $F(tx) = tF(x)$ for all $x \in \mathbb{R}^3$. We define the **induced isometry** $F : \mathbb{P}^2 \to \mathbb{P}^2$ by

$$F\big([x]\big) = \big[F(x)\big].$$

3.15 Theorem: *Every isometry on $\mathbb{P}^2$ is (induced by) a rotation $R_{p,\theta}$ for some $p \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$. The set of isometries on $\mathbb{P}^2$ can be identified with*

$$SO(3, \mathbb{R}) = \big\{A \in M_3(\mathbb{R}) \big| A^T A = I \,,\, \det A = 1\big\}.$$

Proof: Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be an isometry on $\mathbb{P}^2$. Choose $u_1, u_2, u_3 \in \mathbb{S}^2$ so that we have $F([e_1]) = [u_1]$, $F([e_2]) = [u_2]$ and $F([e_3]) = [u_3]$. Then for all k, l we have

$$|u_k \cdot u_l| = \cos d_P\big([u_1], [u_2]\big) = \cos d_P\big(F([e_k]), F([e_l])\big) = \cos d_P\big([e_k], [e_l]\big) = |e_k \cdot e_l| = \delta_{k,l}.$$

When $k \neq l$ we have $|u_k \cdot u_l| = 0$ so that $u_k \cdot u_l = 0$, and when $k = l$ we have $|u_k \cdot u_l| = 1$ and $u_k \cdot u_l = u_k \cdot u_k \geq 0$ so that $u_k \cdot u_l = 1$. Thus $u_k \cdot u_l = \delta_{k,l}$ for all k, l so that $\{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3) \in \mathbb{S}^2$ (so that $[x]$ is an arbitrary element in $\mathbb{P}^2$. Choose $y \in \mathbb{S}^2$ so that $f([x]) = [y]$. For each index k we have

$$|y \cdot u_k| = \cos d_P\big([y], [u_k]\big) = \cos d_P\big(F[x], F[e_k]\big) = \cos d_P\big([x], [e_k]\big) = |x_k \cdot e_k| = |x_k|.$$

Since $\{u_1, u_2, u_3\}$ is orthonormal, we have

$$y = \sum_{k=1}^{3} (y \cdot u_k) u_k = \sum_{k=1}^{3} \pm x_k u_k = Ax$$

where $A = A(x)$ is one of the 8 matrices $\big(\pm u_1, \pm u_2, \pm u_3\big)$. Thus we have

$$F([x]) = [y] = \big[Ax\big]$$

where $A = A(x)$ is one of the 4 matrices $\big(\pm u_1, \pm u_2, \pm u_3\big)$ with determinant equal to 1.

Finally we remark, without providing a rigorous proof, that every isometry is continuous and that the matrix $A = A(x)$ must be constant for all $x \in \mathbb{S}^2$, otherwise F would not be continuous.

Zero Sets of Polynomials

3.16 Definition: For $(x,y,z)\in\mathbb{R}^3$ with $(x,y,z)\neq(0,0,0)$, we write

$$[x,y,z]=\text{Span}\big\{(x,y,z)\big\}\in\mathbb{P}^2.$$

Let $U_1=\big\{[x,y,z]\big|x\neq 0\big\}$, $U_2=\big\{[x,y,z]\big|y\neq 0\big\}$ and $U_3=\big\{[x,y,z]\big|z\neq 0\big\}$. Note that $\mathbb{P}^2=U_1\cup U_2\cup U_3$. We define three **gnomic projections** $\phi_k:U_k\to\mathbb{R}^2$ by

$$\phi_1\big([x,y,z]\big)=\left(\tfrac{y}{x},\tfrac{z}{x}\right)\ ,\ \phi_2\big([x,y,z]\big)=\left(\tfrac{x}{y},\tfrac{z}{y}\right)\ ,\ \phi_3\big([x,y,z]\big)=\left(\tfrac{x}{z},\tfrac{y}{z}\right).$$

Sometimes we identify U_k with $\mathbb{R}^2$ using ϕ_k, and then we can consider $\mathbb{P}^2$ to be a union of three copies of $\mathbb{R}^2$. For each index k, let L_k be the projective line $L_k=\mathbb{P}^2\setminus U_k$ and note that $\mathbb{P}^2$ is the disjoint union $\mathbb{P}^2=U_k\cup L_k$. When we use the gnomic projection ϕ_k, the line L_k is called the **line at infinity**. When we identify U_k with $\mathbb{R}^2$ using ϕ_k, we consider $\mathbb{P}^2$ to be the disjoint union of $\mathbb{R}^2$ with the line at infinity.

We remark that more generally, given any projective line L in $\mathbb{P}^2$ we can define a gnomic projection $\phi:U=\mathbb{P}^2\setminus L\to\mathbb{R}^2$ as follows: let $u\in\mathbb{S}^2$ be a pole for L, choose $v,w\in\mathbb{S}^2$ so that $\{u,v,w\}$ is an orthonormal basis for $\mathbb{R}^3$, then define $\phi(xu+yv+zw)=\left(\frac{y}{x},\frac{z}{x}\right)$.

3.17 Definition: Given a polynomial $f(x,y)$ in two variables, we define the **zero set** of f in $\mathbb{R}^2$ to be the set

$$Z(f)=\big\{(x,y)\in\mathbb{R}^2\big|f(x,y)=0\big\}\subseteq\mathbb{R}^2.$$

Given a polynomial $F(x,y,z)$ in three variables, we define the **zero set** of F in $\mathbb{R}^3$ to be the set

$$Z(F)=\big\{(x,y,z)\in\mathbb{R}^3\big|F(x,y,z)=0\big\}\subseteq\mathbb{R}^3.$$

3.18 Example: When $f(x,y)=y-x^2$, the zero set $Z(f)$ is the parabola $y=x^2$. When $g(x,y)=y-p(x)$, where $p(x)$ is a polynomial in one variable, the zero set $Z(f)$ is the graph $y=p(x)$. When $h(x,y)=x^2+y^2-1$ the zero set $Z(f)$ is the circle $x^2+y^2=1$. When $F(x,y,z)=x^2+y^2+z^2-1$ we have $Z(F)=\mathbb{S}^2$.

3.19 Definition: A polynomial $F(x,y,z)$ is called **homogeneous** of degree n when for every term $c\,x^iy^jz^k$ appearing in F we have $i+j+k=n$. Notice that when F is homogeneous of degree n we have

$$F(tx,ty,tz)=t^nF(x,y,z)\ \text{ for all }\ t\in\mathbb{R}$$

so that for all $(x,y,z)\in\mathbb{R}^3$ we have

$$(x,y,z)\in Z(F)\Longrightarrow t(x,y,z)\in Z(F)\text{ for all }t\in\mathbb{R}\Longrightarrow[x,y,z]\subseteq Z(F).$$

In this case we define the **zero set** of the homogeneous polynomial F in $\mathbb{P}^2$ to be the set

$$Z(F)=\big\{[x,y,z]\in\mathbb{P}^2\big|F(x,y,z)=0\big\}\subseteq\mathbb{P}^2.$$

We do not distinguish notationally between the zero sets $Z(F)\subseteq\mathbb{R}^3$ and $Z(F)\subseteq\mathbb{P}^2$.

3.20 Exercise: Let $F(x,y,z)=x^2+y^2-z^2$. Draw a picture of $Z(F)$.

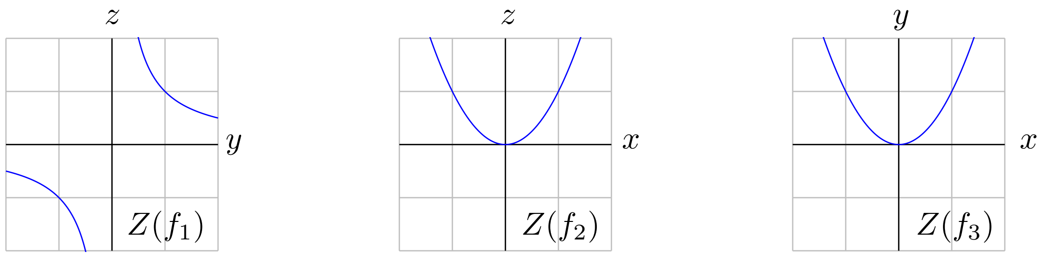
3.21 Definition: Let $F(x,y,z)$ be a homogeneous polynomial. Define

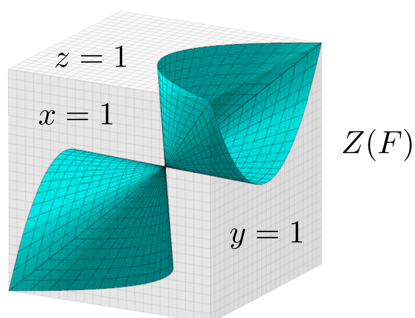
$$f_1(y,z)=F(1,y,z)\ ,\ f_2(x,z)=F(x,1,z)\ \text{ and }\ f_3(x,y)=F(x,y,1).$$

The polynomials f_1, f_2 and f_3 are called the **dehomogenizations** of F.

3.22 Example: Let $F(x,y,z) = yz - x^2$. Use the zero sets $Z(f_1)$, $Z(f_2)$ and $Z(f_3)$ to find the intersection of $Z(F)$ with each of the planes $x = 1$, $y = 1$ and $z = 1$, then draw a picture of $Z(F)$.

Solution: The dehomogenizations are given by $f_1(y,z) = yz - 1$, $f_2(x,z) = z - x^2$ and $f_3(x,y) = y - x^2$. We draw a picture of the zero sets $Z(f_1)$, $Z(f_2)$ and $Z(f_3)$. Then, to draw a picture of $Z(f)$, we draw a cube whose faces are given by $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$, then we draw the zero set $Z(f_1)$ on the face $x = 1$, the zero set $Z(f_2)$ on the face $y = 1$, and the zero set $Z(f_3)$ on the face $z = 1$, in order to obtain a curve on the front, right and top faces of the cube. The points in $Z(f)$ are the points on the lines through 0 which pass through the points on this curve along the surface of the cube.





3.23 Remark: For a homogeneous polynomial $F(x,y,z)$, when we regard $\mathbb{R}^2$ as a subset of $\mathbb{P}^2$ by identifying the point $(x,y) \in \mathbb{R}^2$ with the point $[x,y,1] \in \mathbb{P}^2$, the zero set $Z(f_3) \subseteq \mathbb{R}^2$ is the restriction of the zero set $Z(F) \subseteq \mathbb{P}^2$ to the subset $\mathbb{R}^2 \subseteq \mathbb{P}^2$.

3.24 Definition: Given a polynomial $f(x,y)$ in two variables of degree n, we define the **homogenization** of f to be the homogeneous polynomial $F(x,y,z)$ obtained by replacing each term $c\,x^i y^j$ in f by the term $c\,x^i y^j z^k$ with $k = n - i - j$. Equivalently, we define

$$F(x,y,z) = z^n f\left(\tfrac{x}{z}, \tfrac{y}{z}\right).$$

The zero set $Z(F) \subseteq \mathbb{P}^2$ is called the **projective completion** of the zero set $Z(f) \subseteq \mathbb{R}^2$. The points of the form $[x,y,0] \in \mathbb{P}^2$ which lie in $Z(F)$ are called the **zeros of f at infinity**.

3.25 Remark: When $F(x,y,z)$ is the homogenization of $f(x,y)$, note that f is equal to the dehomogenization f_3 of F, so $Z(f) \subseteq \mathbb{R}^2$ is the restriction of $Z(F) \subseteq \mathbb{P}^2$ to $\mathbb{R}^2 \subseteq \mathbb{P}^2$.

3.26 Example: Let $f(x,y) = xy - 1$. Find the zeros of f at infinity.

Solution: We homogenize to get $F(x,y,z) = xy - z^2$. The zeros of $f(x,y)$ at infinity are the zeros of $F(x,y,z)$ with $z = 0$. We have $F(x,y,0) = xy$ and $xy = 0$ when $x = 0$ or $y = 0$, and so the zeros at infinity are the lines $x = z = 0$ and $y = z = 0$. Using homogeneous coordinates, the zeros at infinity are the points $[0,1,0]$ and $[1,0,0]$.

3.27 Exercise: Let $f(x,y) = y - x^3$. Draw the projective completion of $Z(f)$ and find the zeros of f at infinity.

Conic Sections

3.28 Definition: For $p \in \mathbb{R}^3$, $u \in \mathbb{S}^2$ and $\phi \in \left(0, \frac{\pi}{2}\right)$, the **double cone** with vertex at p and axis in the direction $u \in \mathbb{S}^2$, which makes the angle ϕ with its axis, is the set

$$V = V(p,u,\phi) = \left\{ x \in \mathbb{R}^3 \,\Big|\, \big|(x-p)\cdot u\big| = |x-p|\cos\phi \right\}.$$

The intersection of a double cone in $\mathbb{R}^3$ with the xy-plane is called a **conic section** in $\mathbb{R}^2$. When the vertex of the cone lies in the xy-plane, the intersection can be a point or a line or a pair of intersecting lines, and these are called **degenerate conic sections**.

3.29 Note: The double cone V with vertex $(a,b,c) \in \mathbb{R}^3$ and axis direction $(u,v,w) \in \mathbb{S}^2$ and angle ϕ is given by the equation

$$\Big((x-a, y-b, z-c)\cdot(u,v,w)\Big)^2 = \big((x-a)^2 + (y-b)^2 + (z-c)^2\big)\cos^2\phi.$$

This equation is of degree 2 in x, y and z. The curve of intersection of V with the xy-plane is obtained by setting $z=0$ in the above equation to obtain a degree 2 equation in x and y. Thus every conic section in $\mathbb{R}^2$ is given by a degree 2 equation in x and y.

3.30 Note: There is a theorem in linear algebra which states that every symmetric matrix is orthogonally diagonalizable. It follows from this theorem that the solution set of every degree 2 equation in x and y is either empty, or is a point, a line, a pair of lines, a parabola, a circle, an ellipse, or a hyperbola. By diagonalizing a symmetric matrix, we can find a rotation and a translation of the xy-plane to move the conic section into standard position.

3.31 Example: Diagonalize a symmetric matrix to describe the conic section

$$8x^2 - 12xy + 17y^2 - 36x + 2y = 47.$$

Solution: We can write $8x^2 - 12xy + 17y^2 = (x,y)\,A\begin{pmatrix} x \\ y \end{pmatrix}$ where $A = \begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix}$. The characteristic polynomial of A is

$$f_A(\lambda) = (8-\lambda)(17-\lambda) - 36 = \lambda^2 - 25\lambda + 100 = (\lambda-5)(\lambda-20)$$

so the eigenvalues are $\lambda = 5$ and $\lambda = 20$. Performing row operations gives

$$A - 5I = \begin{pmatrix} 3 & -6 \\ -6 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A - 20I = \begin{pmatrix} -12 & -6 \\ -6 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

and so unit vectors for the eigenvalues 5 and 20 are $\frac{1}{\sqrt{5}}(2,1)^T$ and $\frac{1}{\sqrt{5}}(-1,2)^T$. We change coordinates (scaling by $\sqrt{5}$ and rotating by $\tan^{-1}\frac{1}{2}$) by letting $x = 2u - v$ and $y = u + 2v$. We have

$$\begin{aligned}
& 8x^2 - 12xy + 17y^2 - 36x + 2y = 47 \\
\iff\ & 8(2u-v)^2 - 12(2u-v)(u+2v) + 17(u+2v)^2 - 36(2u-v) + 2(u+2v) = 47 \\
\iff\ & 25\,u^2 + 100\,v^2 - 70\,u + 40\,v = 47 \\
\iff\ & 25\big(u - \tfrac{7}{5}\big)^2 + 100\big(v + \tfrac{1}{5}\big)^2 = 100 \\
\iff\ & \tfrac{(u-\frac{7}{5})^2}{4} + \tfrac{(v+\frac{1}{5})^2}{1} = 1.
\end{aligned}$$

This is the ellipse in the uv-plane centred at $\left(\frac{7}{5}, -\frac{1}{5}\right)$ with vertices $\left(\frac{7}{5}, -\frac{1}{5}\right) \pm (2,0)$ and $\left(\frac{7}{5}, -\frac{1}{5}\right) \pm (0,1)$. Changing back to the coordinates x and y (by rotating and scaling), we obtain the ellipse centred at $(x,y) = (3,1)$ with vertices at $(3,1) \pm (4,2)$ and $(3,1) \pm (-1,2)$.

3.32 Lemma: *Consider the double cone $V(p,u,\phi)$.*
(1) When $p=(0,0,-h)$ with $h>0$ and $u=\frac{1}{\sqrt{2}}(0,1,1)$ and $\phi=\frac{\pi}{4}$, the intersection of $V(p,u,\phi)$ with the xy-plane is the parabola $y=\frac{1}{2h}\,x^2$.
(2) When $p=(0,0,h)$ with $h>0$ and $u=(1,0,0)$ and $\phi\in\left(0,\frac{\pi}{2}\right)$, the intersection of $V(p,u,\phi)$ with the xy-plane is the hyperbola $\frac{x^2}{h^2\cot^2\phi}-\frac{y^2}{h^2}=1$.
(3) When $p=(0,0,h)$ with $h>0$, and $u=(\sin\theta,0,\cos\theta)$ with $\theta\in\left[0,\frac{\pi}{4}\right)$, and $\phi=\frac{\pi}{4}$, the intersection of $V\big(p,u,\phi\big)$ with the xy-plane is the ellipse $\frac{(x+h\tan 2\theta)^2}{h^2\sec^2 2\theta}+\frac{y^2}{h^2\sec 2\theta}=1$.

Proof: To prove Part 1, let $p=(0,0,-h)$ with $h>0$, let $u=\frac{1}{\sqrt{2}}(0,1,1)$, and let $\phi=\frac{\pi}{4}$. The double cone $V(p,u,\phi)$ is given by $\big((x,y,z+h)\cdot\frac{1}{\sqrt{2}}(0,1,1)\big)^2=\big(x^2+y^2+(z+h)^2\big)\big(\frac{1}{\sqrt{2}}\big)^2$, that is $(y+z+h)^2=x^2+y^2+(z+h)^2$. The intersection of $V(p,u,\phi)$ with the xy-plane is given by setting $z=0$ to get $(y+h)^2=x^2+y^2+h^2$, that is $y=\frac{1}{2h}\,x^2$.

To prove Part 2, let $p=(0,0,h)$ with $h>0$, let $u=(1,0,0)$, and let $\phi\in\left(0,\frac{\pi}{2}\right)$. The double cone $V(p,u,\phi)$ is given by $\big((x,y,z-h)\cdot(1,0,0)\big)^2=\big(x^2+y^2+(z-h)^2\big)\cos^2\phi$, that is $x^2=(x^2+y^2+(z-h)^2)\cos^2\phi$. The intersection of $V(p,u,\phi)$ with the xy-plane is given by setting $z=0$ to get $x^2=(x^2+y^2+h^2)\cos^2\phi$, that is $x^2(1-\cos^2\phi)-y^2\cos^2\phi=h^2\cos^2\phi$. Dividing by $h^2\cos^2\phi$ gives $\frac{x^2\sin^2\phi}{h^2\cos^2\phi}-\frac{y^2}{h^2}=1$, that is $\frac{x^2}{h^2\cot^2\phi}-\frac{y^2}{h^2}=1$.

To prove Part 3, let $p=(0,0,h)$ with $h>0$, let $u=(\sin\theta,0,\cos\theta)$ with $\theta\in\left[0,\frac{\pi}{4}\right)$, and let $\phi=\frac{\pi}{4}$. The double cone $V(p,u,\phi)$ is given by

$$\big((x,y,z-h)\cdot(\sin\theta,0,\cos\theta)\big)^2=\big(x^2+y^2+(z-h)^2\big)\cos^2\phi$$

that is

$$\big(x\sin\theta+(z-h)\cos\theta\big)^2=\big(x^2+y^2+(z-h)^2\big)\cdot\tfrac{1}{2}.$$

The intersection of the cone with the xy-plane is given by setting $z=0$ to obtain

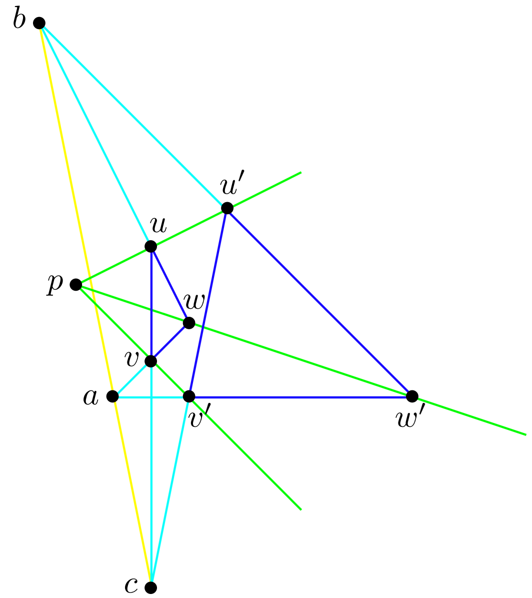
$$\begin{aligned}
(x\sin\theta-h\cos\theta)^2&=(x^2+y^2+h^2)\cdot\tfrac{1}{2}\\
2\big(x^2\sin^2\theta-2\,xh\sin\theta\cos\theta+h^2\cos^2\theta\big)&=x^2+y^2+h^2\\
x^2(1-2\sin^2\theta)+4xh\sin\theta\cos\theta+y^2&=h^2(2\cos^2\theta-1)\\
\cos 2\theta\,x^2+2h\sin 2\theta\,x+y^2&=h^2\cos 2\theta\\
x^2+2h\tan 2\theta\,x+\sec 2\theta\,y^2&=h^2\\
\big(x+h\tan 2\theta\big)^2-h^2\tan^2 2\theta+\sec 2\theta\,y^2&=h^2\\
\big(x+h\tan 2\theta\big)^2+\sec 2\theta\,y^2&=h^2\sec^2 2\theta\\
\frac{(x+h\tan 2\theta)^2}{h^2\sec^2 2\theta}+\frac{y^2}{h^2\sec 2\theta}&=1
\end{aligned}$$

3.33 Corollary: *Given any curve C in the xy-plane which is a point, a line, a pair of intersecting lines, a parabola, a circle, an ellipse, or a hyperbola, one can find a double cone in $\mathbb{R}^3$ whose intersection with the xy-plane is equal to the given curve.*

3.34 Definition: A **conic** in $\mathbb{P}^2$ is the zero set of a homogenous polynomial of degree 2 in x, y and z. Note that for at least one of the variables, when we dehomogenize on that variable we obtain a degree 2 polynomial in the other 2 variables, so a conic in $\mathbb{P}^2$ is the projective completion of a conic section in $\mathbb{R}^2$. The completion of the empty set, a point, a line or a pair of lines is a degenerate conic, and the completion of a parabola, a circle, an ellipse, or a hyperbola is a non-degenerate conic.

3.35 Theorem: *(Desargue's Theorem) Let* u*,* v*,* w*,* u'*,* v' *and* w' *be distinct points in* $\mathbb{P}^2$ *with* u*,* v *and* w *noncolinear and with* u'*,* v' *and* w' *noncolinear. Suppose that the line* u, u'*, the line* v, v' *and the line* w, w' *all intersect at a point* p*. Let* a *be the point of intersection of lines* v, w *and* $v'w'$*, let* b *be the point of intersection of lines* w, u *and* $w'u'$*, and let* c *be the point of intersection of lines* u, v *and* $u'v'$*. Then the points* a*,* b *and* c *are colinear.*

Proof: Choose a (projective) line M in $\mathbb{P}^2$ which does not pass through any of the given points, and use a gnomic projection from $\mathbb{P}^2 \setminus M$ to $\mathbb{R}^2$ to project all of the given points and lines to corresponding points and lines in the xy-plane. Use the same variables $u, v, w, \cdots$ to denote the corresponding points in the xy-plane. Then $[u, v, w]$ and $[u', v', w']$ are two Euclidean triangles in the xy-plane, and the three lines containing line segments $[u, u']$, $[v, v']$ and $[w, w']$ all intersect at the point p, as shown in the diagram below. Raise the points w and w' vertically out of the xy-plane so that the line in $\mathbb{R}^3$ through w and w' still passes through the point p (which is still in the xy-plane). The diagram remains unchanged when we are looking at the xy-plane with the z-axis pointing towards us, but now the triangles $[u, v, w]$ and $[u', v', w']$ are no longer contained in the xy-plane. Let P and P' be the planes in $\mathbb{R}^3$ which contain $[u, v, w]$ and $[u'v'w']$. Note that the planes P and P' are not equal because, for example, we have $u \in P$ but $u \notin P'$ (the intersection of the plane P' with the xy-plane is the line containing $[u', v']$, which does not contain u). Since the planes P and P' are not equal, they intersect in a line L in $\mathbb{R}^3$. Since the line through $[v, w]$ is contained in P we have $a \in P$, and since the line through $[v', w']$ is contained in P' we have $a \in P'$, and so we have $a \in P \cap P' = L$. Similarly, $b \in L$ and $c \in L$.

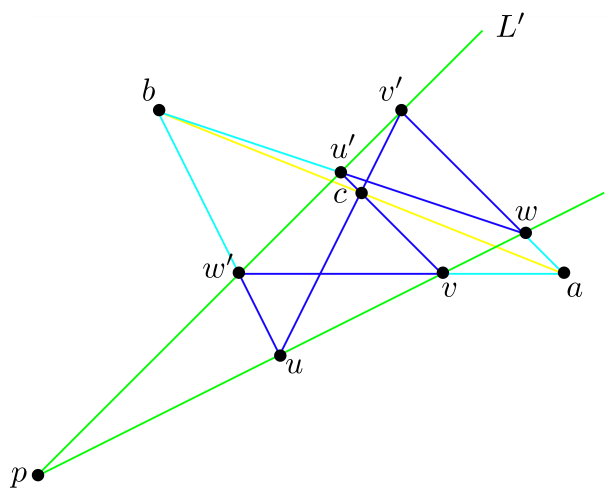


3.36 Theorem: *(Pappus' Theorem) Let L and L' be two distinct lines in $\mathbb{P}^2$. Let u, v and w be points on L and let u', v' and w' be points on L' with all 6 points distinct. Let a be the point of intersection of lines v,w' and w,v', let b be the point of intersection of lines w,u' and u,w', and let c be the point of intersection of lines u,v' and v,u'. Then the points a, b and c are colinear.*

Proof: Let p be the point of intersection of L and L'. Choose a line M in $\mathbb{P}^2$ which does not pass through any of the points p,u,v,w,u',v',w',a,b,c and use a gnomic projection from $\mathbb{P}^2 \setminus M$ to $\mathbb{R}^2$ to project all of the points and lines to corresponding points and lines in the xy-plane. Use the same variables to denote the corresponding points and lines in the xy-plane. Note that we can use a translation to send p to the origin, and then we can use a linear map to send the line L to the x-axis and the line L' to the y-axis, so we may assume that $p=(0,0)$, $u=(r,0)$, $v=(s,0)$, $w=(t,0)$, $u'=(0,k)$, $v'=(0,\ell)$ and $w'=(0,m)$. It is then straightforward (but tedious) to calculate the coordinates of the points a, b and c, and to verify that they are colinear. Here are some of the steps. The line v,w' has equation $y=m-\frac{m}{s}\,x$, or $mx+sy=ms$. The line w,v' has equation $y=\ell-\frac{\ell}{t}\,x$, or $\ell x+ty=\ell t$. The point of intersection of these two lines is $a=\frac{1}{mt-\ell s}\big(st(m-\ell),\ell m(t-s)\big)$. Similarly, we have $b=\frac{1}{kr-mt}\big(rt(k-m),km(r-t)\big)$ and $c=\frac{1}{\ell s-kr}\big(sr(\ell-k),kl(s-r)\big)$. Verify that three points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are colinear when $(y_2-y_1)(x_3-x_1)=(y_3-y_1)(x_2-x_1)$. In particular, the points a, b and c are colinear when

$$\left(\tfrac{km(r-t)}{kr-mt}-\tfrac{\ell m(t-s)}{mt-\ell s}\right)\left(\tfrac{sr(\ell-k)}{\ell s-kr}-\tfrac{st(m-\ell)}{mt-\ell s}\right)=\left(\tfrac{k\ell(s-r)}{\ell s-kr}-\tfrac{\ell m(t-s)}{mt-\ell s}\right)\left(\tfrac{rt(k-m)}{kr-mt}-\tfrac{st(m-\ell)}{mt-\ell s}\right).$$

By multiplying both sides by $(kr-mt)(mt-\ell s)(\ell s-kr)(mt-\ell s)$ then expanding both sides (which is tedious), one finds that equality does hold and so a, b and c are colinear.

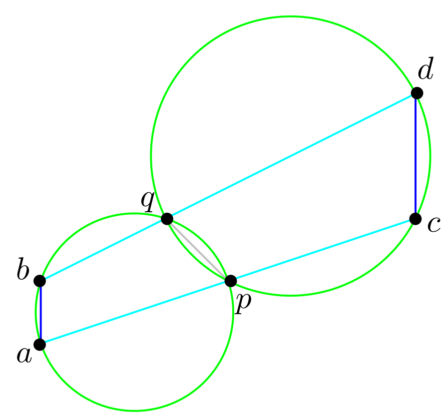


3.37 Note: Let us pause to mention some useful facts about geometry in $\mathbb{R}^2$. Note that given three points $a,b,c\in\mathbb{R}^2$, the points are colinear when $\angle_o abc = 0$ or $\pi \pmod{2\pi}$, that is when $2\angle_o abc = 0 \pmod{2\pi}$. Given four points $a,b,c,d\in\mathbb{R}^2$, note that $[a,b]$ and $[c,d]$ are parallel when $\angle_o cab = \angle_o acd$ or $\angle_o cab = \angle_o acd + \pi \pmod{2\pi}$, that is when $2\angle_o cab = 2\angle_o acd \pmod{2\pi}$. We also recall that given four points a,b,c,p, we can add oriented angles to get $\angle_o apb + \angle_o bpc = \angle_o apc \pmod{2\pi}$. Let us use these facts to prove a pleasing lemma (which we use to prove Pascal's Theorem, below).

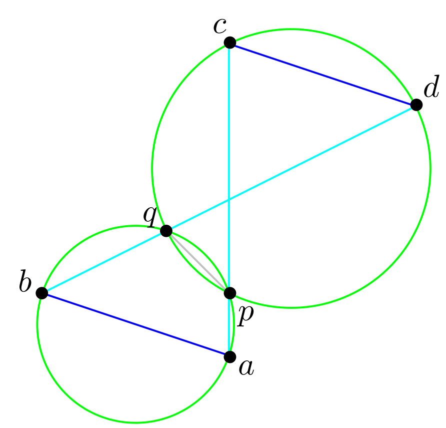
3.38 Lemma: *Let C and D be two circles in $\mathbb{R}^2$ which intersect at p and q. Let $a,b\in C$. Suppose line a,p meets D at c and line b,q meets D at d with all the points a,b,c,d,p,q distinct. Then $[a,b]$ is parallel to $[c,d]$.*

Proof: Working modulo 2π, we have

$$\begin{aligned}
2\,\angle_o cab &= 2\big(\angle_o cap + \angle_o pab\big) &&\text{, by adding angles}\\
&= 2\,\angle_o pab &&\text{, since } c,a,p \text{ are colinear (so } 2\angle_o cap = 0)\\
&= 2\,\angle_o pqb &&\text{, these angles are subtended by the chord } [p,b] \text{ in } C\\
&= 2\big(\angle_o pqd + \angle_o dqb\big) &&\text{, by adding angles}\\
&= 2\,\angle_o pqd &&\text{, since } p,q,b \text{ are colinear}\\
&= 2\,\angle_o pcd &&\text{, these angles are subtended by the chord } [p,d] \text{ in } D\\
&= 2\big(\angle_o pca + \angle_o acd\big) &&\text{by adding angles}\\
&= 2\,\angle_o acd &&\text{, since } p,c,a \text{ are colinear.}
\end{aligned}$$



$\angle_o cab = \pi + \angle_o acd$

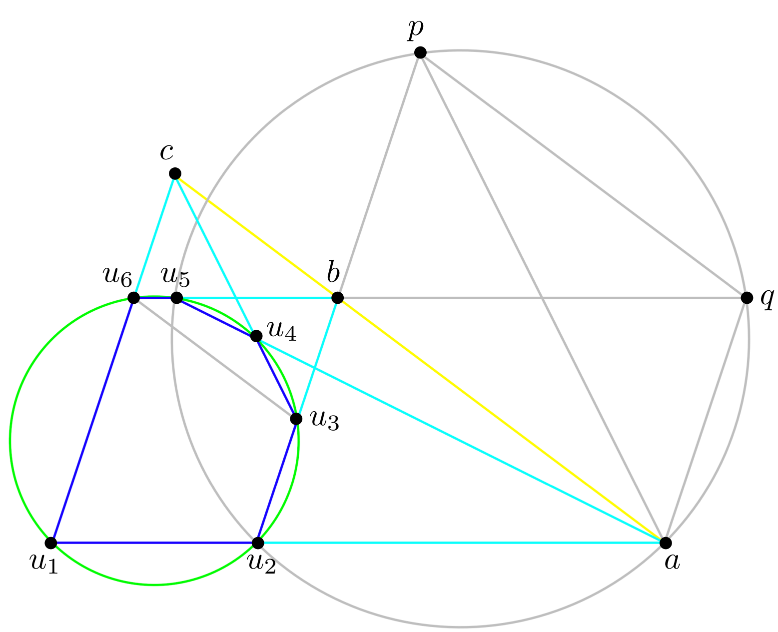


$\angle_o cab = \angle_o acd$

3.39 Theorem: *(Pascal's Theorem) Let C be a conic in $\mathbb{P}^2$. Let $u_1, u_2, \cdots, u_6$ be six distinct points on C. Let a be the point of intersection of lines u_1, u_2 and u_4, u_5, let b be the point of intersection of lines u_2, u_3 and u_5, u_6, and let c be the point of intersection of lines u_3, u_4 and u_6, u_1. Then a, b and c are colinear.*

Proof: The case in which C is empty, or C is a single point cannot occur, the case in which C is a single line is obvious, and the case that C is a pair of lines is Pappus' Theorem. Suppose that C is a non-degenerate conic. Choose a projective line M which does not pass through any of the points u_k, a, b, c, and use a gnomic projection from $\mathbb{P}^2 \setminus M$ to $\mathbb{R}^2$ to project all of the points and lines to corresponding points and lines in $\mathbb{R}^2$. Use the same variables to represent the corresponding curve and points in $\mathbb{R}^2$. The new curve C is a parabola, a circle, an ellipse or a hyperbola in $\mathbb{R}^2$. Choose a double cone V in $\mathbb{R}^3$ whose intersection with the xy-plane is equal to the curve C. Consider the projective space centred at the vertex p of the cone, consisting of all the lines in $\mathbb{R}^3$ through p. Replace C by its projective completion in the projective space centred at p, which consists of all the lines through p in V. Use another gnomic projection, this time from p to a plane which is perpendicular to the axis of the double cone V so that C is replaced by a circle. At this stage, C is a circle in $\mathbb{R}^2$ and $u_1, \cdots, u_6$ are six distinct points on C, the (Euclidean) line through $[u_1, u_2]$ intersects the line through $[u_4, u_5]$ at a, the line through $[u_2, u_3]$ intersects the line through $[u_5, u_6]$ at b, and the line through $[u_3.u_4]$ intersects the line through $[u_6, u_1]$ at c. We need to show that a, b and c are colinear in $\mathbb{R}^2$.

Let D be the circle through u_2, u_5 and a. Say the line u_2, u_3 meets D at u_2 and p, and say the line u_5, u_6 meets D at u_5 and q. By the above lemma, $[u_1, u_6]$ is parallel to $[a, q]$, and $[u_3, u_4]$ is parallel to $[p, a]$, and $[u_3, u_6]$ is parallel to $[p, q]$, so the edges of triangle $[u_3, u_6, c]$ are parallel to the corresponding edges of triangle $[p, a, q]$, and also, the edges of triangle $[u_3, u_6, b]$ are parallel to the corresponding edges of $[p, q, b]$. It follows that $[u_3, u_6, c]$ is similar to $[p, q, a]$ with scaling ratio $r = \frac{|u_3 - u_6|}{|p-q|}$, and that $[u_3, u_6, b]$ is similar to $[p, q, b]$ with the same scaling ratio r. By scaling by the factor r and applying the Side Angle Side Theorem (Corollary 1.54) it follows that $[u_6, b, c]$ is similar to $[q, b, a]$, because the angle at u_6 in $[u_6, b, c]$ is equal to the angle at q in $[q, b, a]$, and the adjacent sides are scaled by $r = \frac{|q-b}{|u_6-b|} = \frac{|q-a|}{|u_6-c|}$. Because u_6, b, q are colinear, and the angle at b in $[u_6, b, c]$ is equal to the angle at b in $[q, b, a]$, it follows that c, b, a are colinear.



3.40 Remark: The Fundamental Theorem of Algebra states that every polynomial $p(x)$ in one variable can be factored over the complex numbers into linear factors, so that if $p(x)$ is of degree n then we have $p(x) = c(x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_l)^{m_l}$ for some distinct $a_i \in \mathbb{C}$ with $m_1 + m_2 + \cdots + m_l = n$. The numbers $a_i \in \mathbb{C}$ are called the **roots** of p and, for each index i, the positive integer m_i is called the **multiplicity** of the root a_i. Thus every polynomial of degree n has exactly n roots in $\mathbb{C}$ provided that the roots are counted with multiplicity (so the root a_i is counted m_i times). Geometrically, the Fundamental Theorem of Algebra states that given any polynomial p of degree n, the curve $y = p(x)$ intersects the line $y = 0$ at exactly n points, provided the points are allowed to be complex and are counted with multiplicity.

There is a very nice generalization of the Fundamental Theorem of Algebra, called Bézout's Theorem, which states that, given any two polynomials $f(x, y)$ and $g(x, y)$, with no common factors, of degrees m and n, the zero sets $Z(f)$ and $Z(g)$ intersect at exactly nm points, provided that the points are allowed to be complex and are counted with multiplicity, and we also count points at infinity. We shall not prove Bézout's Theorem here, and in fact we shall not even describe precisely what is meant by the term"multiplicity".