Chapter 2. Spherical Geometry

The Sphere, Spherical Distance, and Spherical Circles and Lines

2.1 Definition: The (unit) **sphere** is defined to be the set

$$\mathbb{S}^2 = \big\{u \in \mathbb{R}^3 \big| |u| = 1\big\}.$$

If $u \in \mathbb{S}^2$ then we also have $-u \in \mathbb{S}^2$, and we say that the two points $\pm u$ are **antipodal**. Given points $u, v \in \mathbb{S}^2$, we define the (spherical) **distance** between u and v to be

$$d_S(u,v) = \theta(u,v) = \cos^{-1}(u \cdot v).$$

Note that $0 \le d_S(u,v) \le \pi$ with $d_S(u,v) = 0$ when $u = v$ and $d_S(u,v) = \pi$ when $v = -u$.

2.2 Theorem: *(Euclidean and Spherical Distance) Given points $u, v \in \mathbb{S}^2$, the spherical distance $d_S(u,v)$ determines and is determined by the Euclidean distance $d_E(u,v)$.*

Proof: First we note that since $u, v \in \mathbb{S}^2$ we have $d_S(u,v) \in [0,\pi]$ and $d_E(u,v) \in [0,2]$. Applying the Law of Cosines to the triangle in $\mathbb{R}^3$ with vertices at 0, u and v gives

$$d_E(u,v)^2 = 2 - 2\cos\theta(u,v) = 2 - 2\cos d_S(u,v).$$

Thus we have

$$\begin{aligned} d_E(u,v) &= \sqrt{2 - 2\cos d_S(u,v)} \text{ and} \\ d_S(u,v) &= \cos^{-1}\big(1 - \tfrac{1}{2} d_E(u,v)^2\big). \end{aligned}$$

2.3 Theorem: *(Metric Properties of Spherical Distance) For all $u, v, w \in \mathbb{S}^2$ we have*
(1) (Positive Definiteness) $d_S(u,v) \in [0,\pi]$ with $d_S(u,v) = 0$ if and only if $u = v$ and $d_S(u,v) = \pi$ if and only if $u = -v$,
(2) (Symmetry) $d_S(u,v) = d_S(v,u)$, and
(3) (Triangle Inequality) $d_S(u,w) \le d_S(u,v) + d_S(v,w)$.

Proof: We leave the proofs of Parts (1) and (2) as an exercise. To prove Part (3), note that

$$\begin{aligned} \cos\big(\theta(u,v) + \theta(v,w)\big) &= \cos\theta(u,v)\cos\theta(v,w) - \sin\theta(u,v)\sin\theta(v,w) \\ &= (u \cdot v)(v \cdot w) - |u \times v||v \times w| \\ &\le (u \cdot v)(v \cdot w) - (u \times v) \cdot (v \times w) \\ &= (u \cdot v)(v \cdot w) - \big((u \cdot v)(v \cdot w) - (u \cdot w)(v \cdot v)\big) \\ &= u \cdot w = \cos\theta(u,w). \end{aligned}$$

Since $\cos\theta$ is decreasing with θ, it follows that $\theta(u,w) \le \theta(u,v) + \theta(v,w)$.

2.4 Theorem: *(Spherical Area) Given two parallel planes which intersect with $\mathbb{S}^2$, the area of the portion of $\mathbb{S}^2$ which lies between the two planes is equal to $2\pi\Delta$ where Δ is the distance between the two planes. In particular, the total area of $\mathbb{S}^2$ is equal to 4π.*

Proof: Rotate the sphere about the origin so that the two parallel planes have equations $x = a$ and $x = b$ with $-1 \le a \le b \le 1$ and note that the distance between the two planes is $\Delta = b - a$. Recall, from calculus, that the area of the surface which is obtained by revolving the graph of $z = f(x)$ for $a \le x \le b$ about the x-axis is equal to

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

We apply this formula with $f(x)=\sqrt{1-x^2}$ and $f'(x)=\frac{-x}{\sqrt{1-x^2}}$ to get

$$A=\int_a^b 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx=\int_a^b 2\pi\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}\,dx=\int_a^b 2\pi\,dx=2\pi(b-a).$$

2.5 Definition: For $u\in\mathbb{S}^2$ and $r\in[0,\pi]$, the (spherical) **circle** of radius r centred at u, and the (closed spherical) **disc** of radius r centred at u are the sets

$$\begin{aligned}C(u,r)&=\big\{x\in\mathbb{S}^2\big|d_S(x,u)=r\big\}\text{ and}\\ D(u,r)&=\big\{x\in\mathbb{S}^2\big|d_S(x,u)\le r\big\}.\end{aligned}$$

Note that when $r=0$ we have $C(u,r)=\{u\}$ and $D(u,r)=\{u\}$ and when $r=\pi$ we have $C(u,r)=\{-u\}$ and $D(u,r)=\mathbb{S}^2$.

2.6 Theorem: *(Spherical Circles) A circle in $\mathbb{S}^2$ is a set of the form $C=\mathbb{S}^2\cap P$ where P is a plane in $\mathbb{R}^3$ whose Euclidean distance from the origin is at most 1.*

Proof: For $u\in\mathbb{S}^2$ and $r\in[0,\pi]$ we have

$$\begin{aligned}C(u,r)&=\big\{x\in\mathbb{S}^2\big|d_S(x,u)=r\big\}=\big\{x\in\mathbb{S}^2\big|\cos d_S(x,u)=\cos r\big\}\\&=\big\{x\in\mathbb{S}^2\big|x\cdot u=\cos r\big\}=\mathbb{S}^2\cap P\end{aligned}$$

where $P=\big\{x\in\mathbb{R}^3\big|x\cdot u=\cos r\big\}$, which is the plane in $\mathbb{R}^3$ perpendicular to the vector u whose nearest point to the origin is the point $(\cos r)u$.

2.7 Theorem: *(Circumference and Area of Spherical Circles and Discs) Let $u\in\mathbb{S}^2$ and $r\in[0,\pi]$. Then the circumference L of $C(u,r)$ and the area A of $D(u,r)$ are given by*

$$\begin{aligned}L&=2\pi\sin r\text{ , and}\\ A&=2\pi(1-\cos r).\end{aligned}$$

Proof: Let $P=\big\{x\in\mathbb{R}^3\big|x\cdot u=\cos r\big\}$ so that P is the plane whose nearest point to the origin is $(\cos r)u$. Note that for each point $x\in C(u,r)$, the triangle in $\mathbb{R}^3$ with vertices 0, $(\cos r)u$ and x is a right-angle triangle with its right-angle at $(\cos r)u$ whose angle at 0 is equal to $\theta=\theta(x,u)=\cos^{-1}(x\cdot u)=r$. and whose sides are of length $\cos r$, $\sin r$ and 1. The spherical circle $C(u,r)$ is equal to the Euclidean circle in the plane P centred at the point $(\cos r)u$ of Euclidean radius $\sin r$, and so its circumference is $L=2\pi\sin r$.

Let $Q=\big\{x\in\mathbb{R}^3\big|x\cdot u=1\big\}$ so that Q is the plane whose nearest point to the origin is the point u (Q is the tangent plane to $\mathbb{S}^2$ at the point u). Note that the distance between P and Q is $\Delta=1-\cos r$. The spherical disc $D(u,r)$ is equal to the portion of $\mathbb{S}^2$ which lies between the parallel planes P and Q, and so its area is $A=2\pi\Delta=2\pi(1-\cos r)$.

2.8 Definition: A (spherical) **line** (also called a **great circle**) in $\mathbb{S}^2$ is a spherical circle of the form $L=\mathbb{S}^2\cap P$ for some plane P through the origin. Note that each plane P through the origin has two unit normal vectors $\pm u\in\mathbb{S}^2$, and these two normal vectors are called the **poles** of the line $L=\mathbb{S}^2\cap P$. Given $u\in\mathbb{S}^2$, the line in $\mathbb{S}^2$ with poles $\pm u$ is denoted by L_u, so we have

$$L_u=\big\{x\in\mathbb{S}^2\big|x\cdot u=0\big\}=C\big(u,\tfrac{\pi}{2}\big).$$

Two lines in $\mathbb{S}^2$ are said to be **orthogonal** (or **perpendicular**) when their poles are orthogonal (or equivalently when their associated planes are orthogonal).

2.9 Theorem: *(Properties of Spherical Lines)*

(1) Given $u, v \in \mathbb{S}^2$ with $v \neq \pm u$, there is a unique line in $\mathbb{S}^2$ through u and v.
(2) The intersection of any two distinct lines in $\mathbb{S}^2$ consists of two antipodal points.
(3) Given a point $u \in \mathbb{S}^2$ and a line L in $\mathbb{S}^2$ with $L \neq L_u$, there is a unique line in $\mathbb{S}^2$ which passes through u and is perpendicular to L.
(4) There is a unique line which is perpendicular to any two distinct given lines.

Proof: To prove Part 1, let $u, v \in \mathbb{S}^2$ with $u \neq \pm v$. Since $|u| = |v| = 1$ and $u \neq \pm v$, it follows that $\{u, v\}$ is linearly independent, so there is a unique plane P through 0 which contains both u and v, namely $P = \text{Span}\{u, v\}$, so there is a unique (spherical) line in $\mathbb{S}^2$ which contains u and v, namely $L = \mathbb{S}^2 \cap P$. We note that $L = L_w$ with $w = \pm \frac{u \times v}{|u \times v|}$.

To prove Part 2, let L and M be two distinct (spherical) lines in $\mathbb{S}^2$, say $L = \mathbb{S}^2 \cap P$ and $M = \mathbb{S}^2 \cap Q$ where P and Q are distinct planes through 0. The two planes intersect in a line N though 0, and N intersects the unit sphere at two antipodal points, say $\pm w$, and we have $L \cap M = \mathbb{S}^2 \cap P \cap Q = \mathbb{S}^2 \cap N = \{\pm w\}$. Note that if $L = L_u$ and $M = L_v$ so that u and v are unit normal vectors for P and Q, then $u \times v \in P \cap Q = N$ and so $L \cap M = \{\pm w\}$ where $w = \frac{u \times v}{|u \times v|}$.

To prove Part 3, let $u \in \mathbb{S}^2$ and let L be a line in $\mathbb{S}^2$ with $L \neq L_u$, say $L = L_v$ with $v \neq \pm u$. We have $L = \mathbb{S}^2 \cap P$ where P is the plane through 0 with unit normal vector v. The planes Q through 0 which are perpendicular to P are the planes through 0 which pass through v, and so the lines $M = \mathbb{S}^2 \cap Q$ which are perpendicular to $L = \mathbb{S}^2 \cap P = L_v$ are the lines in $\mathbb{S}^2$ through v. Thus the (unique) line which is perpendicular to L and passes through u is the same as the (unique) line through v and u (which exists and is unique by Part a). We note that this unique line is the line L_w with $w = \frac{u \times v}{|u \times v|}$.

To prove Part 4, let L and M be two distinct lines, say $L = L_u$ and $M = L_v$ where $u, v \in \mathbb{S}^2$ with $u \neq \pm v$. As is our proof of Part 3, the lines in $\mathbb{S}^2$ which are perpendicular to $L = L_u$ are precisely the lines in $\mathbb{S}^2$ through u, and the lines in $\mathbb{S}^2$ which are perpendicular to $M = L_v$ are precisely the lines in $\mathbb{S}^2$ through v, and so the (unique) line in $\mathbb{S}^2$ perpendicular to both $L = L_u$ and $M = L_v$ is the (unique) line in $\mathbb{S}^2$ through u and v. We note that the unique line perpendicular to $L = L_u$ and $M = L_v$ is the line $N = L_w$ where $w = \frac{u \times v}{|u \times v|}$.

Oriented Angles

2.10 Definition: For $u \in \mathbb{S}^2$, the **tangent space** to $\mathbb{S}^2$ at u is the 2-dimensional vector space T_u which is orthogonal to u, that is the space

$$T_u = \big\{ x \in \mathbb{R}^3 \big| x \cdot u = 0 \big\}.$$

2.11 Definition: For $u \in \mathbb{S}^2$ and $0 \neq v, w \in T_u$, we define the **oriented angle** from v to w to be the angle $\theta_o(v,w) \in [0, 2\pi)$ from v counterclockwise to w when looking at the plane T_u with the vector u pointing towards us. The (unoriented) **angle** between u and v is the same as the (unoriented) angle $\theta(u,v) = \cos^{-1} \frac{u \cdot v}{|u|\,|v|}$ between u and v in $\mathbb{R}^3$. We have $\theta(u,v) = \theta_o(u,v)$ if $\theta_o(u,v) \in [0,\pi]$, and $\theta(u,v) = 2\pi - \theta_o(u,v)$ if $\theta_o(u,v) \in [\pi, 2\pi)$.

2.12 Theorem: *(Angle Formula) Let $u \in \mathbb{S}^2$ and let $0 \neq v, w \in T_u$. Then*

$$\cos \theta_o(v,w) = \frac{v \cdot w}{|v||w|} \text{ and}$$
$$\sin \theta_o(v,w) = \frac{\det(u,v,w)}{|v||w|}.$$

Proof: When $v \times w = 0$ so that $\{v, w\}$ is linearly dependent, if $w = tv$ with $t > 0$ then $\theta_o(v,w) = \theta(v,w) = 0$ and if $w = tv$ with $t < 0$ then $\theta_o(v,w) = \theta(v,w) = \pi$, and in both cases we have $\sin \theta_o(v,w) = \det(u,v,w) = 0$.

Suppose that $v \times w \neq 0$. Since $v, w \in T_u$ so that v and w are orthogonal to u it follows that the vector $v \times w$ points either in the direction of u or in the direction of $-u$. When $\theta_o(v,w) \in (0,\pi)$, the Right-Hand Rule implies that $v \times w$ points in the direction of u so that $\theta(u, v \times w) = 0$, and in this case we have $\theta_o(v,w) = \theta(v,w)$. When $\theta_o(v,w) \in (\pi, 2\pi)$, the Right-Hand Rule implies that $v \times w$ points in the direction of $-u$ so that $\theta(u, v \times w) = \pi$, and in this case we have $\theta_o(v,w) = 2\pi - \theta(v,w)$. Note that

$$\cos \theta(u, v \times w) = \frac{u \cdot (v \times w)}{|u||v \times w|} = \frac{\det(u,v,w)}{|v \times w|} \text{ so that}$$
$$\det(u,v,w) = |v \times w| \cos \theta(u, v \times w) = \begin{cases} |v \times w| & \text{if } \theta_o(v,w) \in (0,\pi), \\ -|v \times w| & \text{if } \theta_o(v,w) \in (\pi, 2\pi). \end{cases}$$

In the case that $\theta_o(v,w) \in (0,\pi)$ we have

$$\cos \theta_o(v,w) = \cos \theta(v,w) = \frac{v \cdot w}{|v||w|} \text{ , and}$$
$$\sin \theta_o(v,w) = \sin \theta(v,w) = \frac{|v \times w|}{|v||w|} = \frac{\det(u,v,w)}{|v||w|}$$

and in the case that $\theta_o(v,w) \in (\pi, 2\pi)$ we have

$$\cos \theta_o(v,w) = \cos\big(2\pi - \theta(v,w)\big) = \cos \theta(v,w) = \frac{v \cdot w}{|v||w|} \text{ , and}$$
$$\sin \theta_o(v,w) = \sin\big(2\pi - \theta(v,w)\big) = -\sin \theta(v,w) = \frac{-|v \times w|}{|v||w|} = \frac{\det(u,v,w)}{|v||w|}.$$

2.13 Remark: Definition 2.11 is not actually a rigorous mathematical definition, and the proof of the above theorem is not actually a rigorous mathematical proof (since the words "counterclockwise" and "right-hand rule" are not rigorously defined). To be rigorous, we would simply take the formulas in Theorem 2.12 as our definition for the oriented angle, and then the above so-called proof can be taken as an informal motivation for the definition.

2.14 Definition: Let $u, v \in \mathbb{S}^2$ with $v \neq \pm u$. The line in $\mathbb{S}^2$ through u and v is cut, at the points u and v, into two arcs from u to v. The shorter of the two arcs is called the (spherical) **line segment** from u to v and is denoted by $[u, v]$. Note that

$$[u,v] = \left\{x \in \mathbb{S}^2 \middle| x = su + tv \text{ for some } 0 \leq s, t \in \mathbb{R}\right\}.$$

The **vector from** u to v, denoted by u_v, is the unit tangent vector to the arc $[u, v]$ at the point u. By the Right Hand Rule, applied twice, we see that u_v is the unit vector in the direction of the vector $(u \times v) \times u$. Note that since $(u \times v) \cdot u = 0$ so that $\theta(u \times v, u) = \frac{\pi}{2}$, we have $\left|(u \times v) \times u\right| = |u \times v||u| \sin\theta(u \times v, u) = |u \times v|$, and so

$$u_v = \frac{(u \times v) \times u}{|u \times v|} = \frac{v - (u \cdot v)u}{|u \times v|}.$$

Given $u, v, w \in \mathbb{S}^2$ with $v \neq \pm u$ and $w \neq \pm u$, we define the **oriented angle** $\angle_o vuw$ to be

$$\angle_o vuw = \theta_o(u_v, u_w).$$

The (unoriented) **angle** $\angle vuw$ is given by $\angle vuw = \theta(u_v, u_w)$. We have $\angle vuw = \angle_o vuw$ when $\angle_o vuw \in [0, \pi]$, and $\angle vuw = 2\pi - \angle_o vuw$ when $\angle_o vuw \in [\pi, 2\pi)$.

2.15 Theorem: *(Angle Formula) Let $u, v, w \in \mathbb{S}^2$ with $v \neq \pm u$ and $w \neq \pm u$, and let $\alpha_o = \angle_o vuw$. Then*

$$\cos\alpha_o = \frac{(u \times v) \cdot (u \times w)}{|u \times v||u \times w|} = \frac{(v \cdot w) - (u \cdot v)(u \cdot w)}{|u \times v||u \times w|} \text{ and}$$
$$\sin\alpha_o = \frac{\det(u, v, w)}{|u \times v||u \times w|}.$$

Proof: We have

$$\begin{aligned}\cos\alpha_o = \cos\theta_o(u_v, u_w) = u_v \cdot u_w &= \frac{\big(v - (u \cdot v)u\big) \cdot \big(w - (u \cdot w)u\big)}{|u \times v||u \times w|} \\ &= \frac{(v \cdot w) - (u \cdot v)(u \cdot w)}{|u \times v||u \times w|} = \frac{(u \times v) \cdot (u \times w)}{|u \times v||u \times w|}\end{aligned}$$

and

$$\begin{aligned}\sin\alpha_o = \sin\theta_o(u_v, u_w) = \det(u, u_v, u_w) &= \frac{u \cdot \Big(\big(v - (u \cdot v)u\big) \times \big(w - (u \cdot w)u\big)\Big)}{|u \times v||u \times w|} \\ &= \frac{u \cdot \big(v \times w - (u \cdot w)\, v \times u - (u \cdot v)\, u \times w\big)}{|u \times v||u \times w|} = \frac{\det(u, v, w)}{|u \times v||u \times w|}.\end{aligned}$$

Spherical Triangles

2.16 Definition: A (non-degenerate spherical) **triangle** in $\mathbb{S}^2$ is determined by three non-colinear points $u, v, w \in \mathbb{S}^2$, which we call the **vertices** of the triangle. Requiring that u, v and w are non-colinear in $\mathbb{S}^2$ is equivalent to requiring that u, v and w do not all lie on the same plane through the origin in $\mathbb{R}^3$, or equivalently that $\{u, v, w\}$ is linearly independent, or equivalently that $\det(u, v, w) \neq 0$.

We can think of the triangle with vertices u, v and w in several different ways. For example, we can think of the triangle simply as the set of three points $\{u, v, w\}$, or if we wish we can keep track of the the order of the points and think of the triangle as the ordered triple (u, v, w). Alternatively, since u, v and w are non-colinear, no two of the three points are antipodal and so the **edges** $[u, v]$, $[v, w]$ and $[w, u]$ of the triangle are well-defined and, if we want, we can think of the triangle as the union of its three edges. As another alternative, we can think of the triangle as including its interior points, that is we can consider the triangle to be the **solid triangle**

$$[u, v, w] = \left\{x \in \mathbb{S}^2 \middle| x = ru + sv + tw \text{ for some } 0 \leq r, s, t \in \mathbb{R}\right\}.$$

An **ordered triangle** in $\mathbb{S}^2$ consists of an ordered triple (u, v, w) of non-colinear points $u, v, w \in \mathbb{S}^2$, together with the set $[u, v, w]$. When $\det(u, v, w) > 0$, so that $\{u, v, w\}$ is a positively oriented basis for $\mathbb{R}^3$, we say that the triangle $[u, v, w]$ is **positively oriented**, and when $\det(u, v, w) < 0$ we say the triangle $[u, v, w]$ is **negatively oriented**.

2.17 Definition: Given an ordered triangle $[u, v, w]$ in $\mathbb{S}^2$, we shall let a, b and c denote the lengths of the edges $[v, w]$, $[w, u]$ and $[u, v]$ respectively, so that

$$a = d_S(v, w)\ ,\ b = d_S(w, u) \ \text{ and }\ c = d_S(u, v)\,,$$

and we shall let α_o, β_o and γ_o, and α, β and γ be the oriented and unoriented angles

$$\alpha_o = \angle_o vuw\ ,\ \beta_o = \angle_o wvu\ ,\ \gamma_o = \angle_o uwv\ ,\ \alpha = \angle vuw\ ,\ \beta = \angle wvu\ ,\ \gamma = \angle uwv.$$

When $[u, v, w]$ is positively oriented, the angles α_o, β_o and γ_o all lie in the interval $(0, \pi)$ so we have $\alpha = \alpha_0$, $\beta = \beta_o$ and $\gamma = \gamma_o$. When $[u, v, w]$ is negatively oriented, the angles α_o, β_o and γ_o all lie in the interval $(\pi, 2\pi)$, so we have $\alpha = 2\pi - \alpha_o$, $\beta = 2\pi - \beta_o$ and $\gamma = 2\pi - \gamma_o$. In either case, the angles α, β and γ are called the **interior angles** of the triangle, and the angles $2\pi - \alpha$, $2\pi - \beta$ and $2\pi - \gamma$ are called the **exterior angles**.

2.18 Theorem: *(Area of Spherical Triangles) Let $[u, v, w]$ be a positively oriented triangle in $\mathbb{S}^2$ with interior angles α, β and γ. Then the area of $[u, v, w]$ is equal to $(\alpha + \beta + \gamma) - \pi$.*

Proof: Let H_α be the hemisphere which contains u whose boundary is the line through v and w, and let $-H_\alpha$ be the antipodal hemisphere. Define H_β, $-H_\beta$, H_γ and $-H_\gamma$ similarly. Let $W_\alpha = \big(H_\beta \cap H_\gamma\big) \cup \big(-H_\beta \cap -H_\gamma\big)$, and define W_β and W_γ similarly. By looking at the sphere with the vector u pointing towards us, we can see that the double wedge W_α covers $\frac{\alpha}{\pi}$ of the entire sphere and so its area is $A_\alpha = \frac{\alpha}{\pi} \cdot 4\pi = 4\alpha$. Similarly the areas of the double wedges W_β and W_γ are equal to $A_\beta = 4\beta$ and $A_\gamma = 4\gamma$. Notice that when we shade each of the double wedges W_α, W_β and W_γ, the triangle $[u, v, w]$ and its antipodal triangle $[-u, -v, -w]$ are each shaded three times while the rest of the sphere is shaded once. It follows that if we let $S = 4\pi$ be the area of the entire sphere and we let T be the area of the triangle $[u, v, w]$ (which is equal to the area of the antipodal triangle $[-u, -v, -w]$) then we have $A_\alpha + A_\beta + A_\gamma = S + 4T$, that is $4\alpha + 4\beta + 4\gamma = 4\pi + 4T$, hence $T = (\alpha + \beta + \gamma) - \pi$.

2.19 Definition: For an ordered triangle $[u,v,w]$, the **polar triangle** of $[u,v,w]$ is the ordered triangle $[u',v',w']$ where

$$u' = \frac{v\times w}{|v\times w|}\ ,\ v' = \frac{w\times u}{|w\times u|}\ \text{ and } w' = \frac{u\times v}{|u\times v|}\,.$$

We denote the side lengths and the angles of the polar triangle $[u',v',w']$ by a', b', c' and α_o', β_o', γ_o', and α', β' and γ'.

2.20 Theorem: *(The Polar Triangle) Let $[u,v,w]$ be a positively oriented triangle in $\mathbb{S}^2$ with polar triangle $[u',v',w']$. Then*
(1) $[u'',v'',w''] = [u,v,w]$,
(2) $[u',v',w']$ is positively oriented,
(3) $a' = \pi-\alpha$, $b' = \pi-\beta$ and $c' = \pi-\gamma$, and
(4) $\alpha' = \pi - a$, $\beta' = \pi - b$ and $\gamma' = \pi - c$.

Proof: We have

$$v'\times w' = \frac{(w\times u)\times(u\times v)}{|w\times u||u\times v|} = \frac{w\cdot(u\times v)\,u - u\cdot(u\times v)\,w}{|w\times u||u\times v|} = \frac{\det(u,v,w)\,u}{|w\times u||u\times v|}.$$

This is a positive multiple of the vector u and so

$$u'' = \frac{v'\times w'}{|v'\times w'|} = u.$$

Similarly, we have $v''=v$ and $w''=w$. This shows that $[u'',v'',w''] = [u,v,w]$.

Now let us determine the orientation of $[u',v',w']$. We have

$$\det(u',v',w') = u'\cdot(v'\times w') = \frac{v\times w}{|v\times w|}\cdot\frac{\det(u,v,w)\,u}{|w\times u||u\times v|} = \frac{\det(u,v,w)^2}{|u\times v||v\times w||w\times u|}$$

which is positive. This shows that $[u',v',w']$ is positively oriented.

Next, let us find the lengths of the edges of the polar triangle. We have

$$\cos a' = v'\cdot w' = \frac{(w\times u)\cdot(u\times v)}{|w\times u||u\times v|} = -\cos\alpha.$$

It follows that $a' = \pi\pm\alpha$. Since $[u,v,w]$ is positively oriented, we have $\alpha\in(0,\pi)$ and so we must have $a' = \pi-\alpha$. Similarly, we have $b' = \pi-\beta$ and $c' = \pi-\gamma$.

Finally, let us calculate the angles of the polar triangle. We have

$$\cos\alpha' = \frac{(u'\times v')\cdot(u'\times w')}{|u'\times v'||u'\times w'|} = w''\cdot(-v'') = -w\cdot v = -\cos a.$$

It follows that $\alpha' = \pi\pm a$. But since $[u',v',w']$ is positively oriented so that $\alpha'\in(0,\pi)$, we must have $\alpha' = \pi-a$. Similarly, $\beta' = \pi-b$ and $\gamma' = \pi-c$.

2.21 Exercise: Determine how Parts 1, 2, 3 and 4 of the above theorem must be modified when $[u,v,w]$ is a negatively oriented triangle in $\mathbb{S}^2$.

2.22 Theorem: *(The Sine Law) For any ordered triangle* $[u,v,w]$ *we have*

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Proof: We have

$$\sin \alpha = \frac{|\det(u,v,w)|}{|u \times v||u \times w|} = \frac{|\det(u,v,w)|}{\sin c \sin b}$$

and similar formulas hold for $\sin \beta$ and $\sin \gamma$, so we obtain

$$|\det(u,v,w)| = \sin \alpha \sin b \sin c = \sin a \sin \beta \sin c = \sin a \sin b \sin \gamma.$$

2.23 Theorem: *(The First Law of Cosines) For any ordered triangle* $[u,v,w]$ *we have*

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}\ ,\ \cos \beta = \frac{\cos b - \cos a \cos c}{\sin a \sin c}\ \text{ and }\ \cos \gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b}\ .$$

Proof: We have

$$\cos \alpha = \frac{(u \times v) \cdot (u \times w)}{|u \times v||u \times w|} = \frac{(v \cdot w) - (u \cdot w)(v \cdot u)}{|u \times v||u \times w|} = \frac{\cos a - \cos b \cos c}{\sin c \sin b}.$$

The other two formulas may be proven similarly.

2.24 Corollary: *(Side-Side-Side and Side-Angle-Side)*

(1) If we know the lengths of the three sides of a triangle then we can find the three angles.
(2) If we know the lengths of two sides and the angle at the common vertex, then we can find the length of the third side (hence also the other two angles).

2.25 Theorem: *(The Second Law of Cosines) For any ordered triangle* $[u,v,w]$ *we have*

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}\ ,\ \cos b = \frac{\cos \beta + \cos \gamma \cos \alpha}{\sin \gamma \sin \alpha}\ \text{ and }\ \cos c = \frac{\cos \gamma + \cos \alpha \cos \beta}{\sin \alpha \sin \beta}.$$

Proof: Suppose that $[u,v,w]$ is positively orented, and let $[u',v',w']$ be its polar triangle. Then we have $a' = \pi - \alpha$ and $\alpha' = \pi - a$ and so on. We apply the First Law of Cosines to the polar triangle to get

$$\cos a = -\cos \alpha' = \frac{-\cos a' + \cos b' \cos c'}{\sin b' \sin c'} = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}.$$

The case in which $[u,v,w]$ is negatively oriented is left as an exercise.

2.26 Corollary: *(Angle-Angle-Angle and Angle-Side-Angle)*

(1) If we know the three angles of a triangle then we can find the lengths of the three sides.
(2) If we know the length of one edge of a triangle and the angles at either end of the edge, then we can find the third angle (hence also the lengths of the other two sides).

Isometries

2.27 Definition: An $n \times n$ matrix $A \in M_n(\mathbb{R})$ is called **orthogonal** when $A^T A = I$ or equivalently, when its columns form an orthonormal basis for $\mathbb{R}^n$. The set of all orthogonal $n \times n$ matrices is denoted by $O_n(\mathbb{R})$. An **orthogonal map** on $\mathbb{R}^n$ is a map $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form $F(x) = Ax$ for some $A \in O_n(\mathbb{R})$.

2.28 Definition: An **isometry** on $\mathbb{S}^2$ is a bijective map $F : \mathbb{S}^2 \to \mathbb{S}^2$ which preserves distance, that is such that for all $u, v \in \mathbb{S}^2$ we have $d_S\big(F(u), F(v)\big) = d_S(u, v)$.

2.29 Theorem: *(Algebraic Classification of Isometries) Every orthogonal map on $\mathbb{R}^3$ restricts to an isometry on $\mathbb{S}^2$, and every isometry on $\mathbb{S}^2$ extends to an orthogonal map on $\mathbb{R}^3$. Thus the group of isometries on $\mathbb{S}^2$ can be identified with the group $O_3(\mathbb{R})$.*

Proof: Let $A \in O_3(\mathbb{R})$ and define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x) = Ax$. Note that F is bijective with inverse given by $F^{-1}(x) = A^{-1}x = A^T x$. Also note that F preserves Euclidean distance because for all $x, y \in \mathbb{R}^3$ we have

$$\begin{aligned} \big|F(x) - F(y)\big|^2 &= |Ax - Ay|^2 = \big|A(x-y)\big|^2 = \big(A(x-y)\big)^T A(x-y) \\ &= (x-y)^T A^T A(x-y) = (x-y)^T I(x-y) = (x-y)^T(x-y) = |x-y|^2. \end{aligned}$$

Since spherical distance is determined by Euclidean distance, it follows that

$$d_S\big(F(u), F(v)\big) = d_S(u, v) \quad \text{for all} \quad u, v \in \mathbb{S}^2$$

and so F restricts to an isometry on $\mathbb{S}^2$.

Now suppose that $F : \mathbb{S}^2 \to \mathbb{S}^2$ is an isometry on $\mathbb{S}^2$. Since Euclidean distance is determined by spherical distance, we have $\big|F(u) - F(v)\big| = |u - v|$ for all $u, v \in \mathbb{S}^2$. By the Polarization Identity, it follows that for all $u, v \in \mathbb{S}^2$

$$\begin{aligned} F(u) \cdot F(v) &= \tfrac{1}{2}\Big(|F(u)|^2 + |F(v)|^2 - \big|F(u) - F(v)\big|^2\Big) = \tfrac{1}{2}\big(1^2 + 1^2 - |u-v|^2\big) \\ &= \tfrac{1}{2}\big(|u|^2 + |v|^2 - |u-v|^2\big) = u \cdot v \end{aligned}$$

and so the map F preserves the dot product between elements of $\mathbb{S}^2$. In particular, if we let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ then we have $F(e_k) \cdot F(e_l) = e_k \cdot e_l = \delta_{kl}$ for all k, l and so the set $\big\{F(e_1), F(e_2), F(e_3)\big\}$ is also an orthonormal basis for $\mathbb{R}^3$.

Given any element $v \in \mathbb{S}^2$, we can write v uniquely in the form $v = \sum_{k=1}^{3} c_k F(e_k)$, and then for each index l we have $v \cdot F(e_l) = \sum_{k=1}^{3} c_k F(e_k) \cdot F(e_l) = \sum_{k=1}^{3} c_k \delta_{kl} = c_l$. Thus for every $v \in \mathbb{S}^2$ we have $v = \sum_{k=1}^{3} \big(v \cdot F(e_k)\big) F(e_k)$. Replacing v by $F(u)$, we see that for all $u = (u_1, u_2, u_3) \in \mathbb{S}^2$ we have

$$F(u) = \sum_{k=1}^{3} \big(F(u) \cdot F(e_k)\big) F(e_k) = \sum_{k=1}^{3} (u \cdot e_k) F(e_k) = \sum_{k=1}^{3} u_k F(e_k).$$

Let A be the orthogonal matrix with columns $F(e_1)$, $F(e_2)$ and $F(e_3)$. Then for all $u \in \mathbb{S}^2$ we have $F(u) = \sum_{k=1}^{3} u_k F(e_k) = Au$. Thus the isometry $F : \mathbb{S}^2 \to \mathbb{S}^2$ extends to the orthogonal map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by $F(x) = Ax$ for all $x \in \mathbb{R}^3$.

2.30 Note: From now on, we shall not distinguish notationally between an orthogonal matrix $A \in O_3(\mathbb{R})$, the corresponding orthogonal map $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A(x) = Ax$, and the corresponding isometry $A : \mathbb{S}^2 \to \mathbb{S}^2$ obtained by restricting the map A to $\mathbb{S}^2$.

2.31 Definition: The **inversion** (or **antipodal map** on $\mathbb{S}^2$ is the isometry $N : \mathbb{S}^2 \to \mathbb{S}^2$ given by $N(x) = -x$. Given $u \in \mathbb{S}^2$, the **reflection** on $\mathbb{S}^2$ in the line L_u is the isometry $F_u : \mathbb{S}^2 \to \mathbb{S}^2$ obtained by restricting the orthogonal reflection $F_u : \mathbb{R}^3 \to \mathbb{R}^3$ on $\mathbb{R}^3$ in the plane $x \cdot u = 0$. When $L = L_u$ we also write $F_u = F_L$. Given $u \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$, the **rotation** on $\mathbb{S}^2$ about u by θ is the map $R_{u,\theta} : \mathbb{S}^2 \to \mathbb{S}^2$ obtained by restricting the rotation $R_{u,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ by the angle θ in the direction of the fingers of the right hand when the thumb is pointing in the direction of the vector u.

2.32 Note: As a matrix, the inversion is given by $N = -I$. Let us describe the maps F_u and $R_{u,\theta}$ as matrices. Given u, choose a unit vector v which is orthogonal to u and then let $w = u \times v$ so that $\{u, v, w\}$ is an orthonormal basis for $\mathbb{R}^3$. Then the rotation $R_{u,\theta}$ is given by $R_{u,\theta}(u) = u$, $R_{u,\theta}(v) = \cos\theta\, v + \sin\theta\, w$ and $R_{u,\theta}(w) = -\sin\theta\, v + \cos\theta\, w$. Thus, as a matrix, we have

$$R_{u,\theta} = PAP^T \quad \text{where} \quad P = (u, v, w) \text{ and } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Similarly, the reflection F_u is given by $F_u(u) = -u$, $F_u(v) = v$ and $F_u(w) = w$ so that, as a matrix, we have

$$F_u = PAP^T \quad \text{where} \quad P = (u, v, w) \text{ and } A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, using the orthogonal projection map given by $\text{Proj}_u(x) = (x{\cdot}u)u$, we have

$$F_u(x) = x - 2\,\text{Proj}_u(x) = x - 2(x{\cdot}u)u = x - 2u(x \cdot u) = x - 2uu^T x = \big(I - 2uu^T\big)x$$

and so, as a matrix, we have

$$F_u = I - 2uu^T.$$

2.33 Theorem: *(The Product of Two Reflections is a Rotation) Let $u, v, w \in \mathbb{S}^2$ with $v, w \in T_u$. Then*

$$F_w F_v = R_{u, 2\theta_o(v,w)}.$$

Proof: Let $x = u \times v$ so that $\{u, v, x\}$ is a positively oriented orthonormal basis for $\mathbb{R}^3$. Let $\theta_o = \theta_o(v, w)$ so that

$$w = \cos\theta_o\, v + \sin\theta_o\, x.$$

Then $F_v(u) = u - 2\, u \cdot v\, v = u$, $F_v(v) = v - 2\, v \cdot v\, v = -v$ and $F_v(x) = x - 2\, x \cdot v\, v = x$ and so

$$\begin{aligned} F_w F_v(u) &= F_w(u) = u - 2\, u \cdot w\, w = u = R_{u,2\theta_o}(u)\,, \\ F_w F_v(v) &= F_w(-v) = -v + 2\, v \cdot w\, w = -v + 2\cos\theta_o(\cos\theta_o v + \sin\theta_o\, x) \\ &= (2\cos\theta_o - 1)v + 2\sin\theta_o\cos\theta_o\, x = \cos 2\theta_o\, v + \sin 2\theta_o\, x = R_{u,2\theta_o}(v), \text{ and} \\ F_w F_v(x) &= F_w(x) = x - 2\, x \cdot w\, w = x - 2\sin\theta_o(\cos\theta_o\, v + \sin\theta_o\, x) \\ &= -2\sin\theta_o\cos\theta_o\, v + (1 - 2\sin\theta_o)x = -\sin 2\theta_o\, v + \cos 2\theta_o\, x = R_{u,2\theta_o}(x). \end{aligned}$$

2.34 Definition: Given two points $a, b \in \mathbb{R}^3$ with $a \neq b$, recall that the **perpendicular bisector** of a and b in $\mathbb{R}^3$ is the plane P in $\mathbb{R}^3$ which passes through the midpoint $\frac{a+b}{2}$ and is orthogonal to the vector $b-a$, that is the plane

$$P = \left\{ x \in \mathbb{R}^3 \middle| \left(x - \tfrac{a+b}{2}\right) \cdot (b-a) = 0 \right\}.$$

Note that (as in the proof of Theorem 1.60) for $x \in \mathbb{R}^3$ we have

$$x \in P \iff d_E(x,a) = d_E(x,b).$$

Let $u, v \in \mathbb{S}^2$ with $u \neq v$ and let P be the perpendicular bisector of u and v in $\mathbb{R}^3$. Note that since $d_E(0,u) = 1 = d_E(0,v)$, we have $0 \in P$. We define the **perpendicular bisector** of u and v in $\mathbb{S}^2$ to be the line $L = \mathbb{S}^2 \cap P$. Since P is the plane in $\mathbb{R}^3$ through 0 orthogonal to the vector $v-u$, it follows that $L = L_w$ where $w = \frac{v-u}{|v-u|}$. Note that since spherical distance is determined by Euclidean distance, for $x \in \mathbb{S}^2$ we have $x \in L \iff d_S(x,u) = d_S(x,v)$.

2.35 Lemma: *(Reflection in Perpendicular Bisector) Let $u, v \in \mathbb{S}^2$ with $u \neq v$, let P be the perpendicular bisector of u and v in $\mathbb{R}^3$, and let F_P be the orthogonal reflection in P. Then*
(1) $F_P(u) = v$ and $F_P(v) = u$, and
(2) $x \in P \iff F_P(x) = x$.

Proof: Let $w = \frac{v-u}{|v-u|}$. Note that the plane P has equation $x \cdot w = 0$ and the orthogonal reflection F_P is given by $F_P(x) = x - 2(x \cdot w)\, w$. We have

$$\begin{aligned} F_P(u) &= u - 2(u \cdot w)\, w = u - \frac{2\, u \cdot (v-u)}{|v-u|^2}(v-u) = u - \frac{2\big(u \cdot v - |u|^2\big)}{|u|^2 - 2(u \cdot v) + |v|^2}(v-u) \\ &= u - \frac{2(u \cdot v - 1)}{1 - 2u \cdot v + 1}(v-u) = u + (v-u) = v \end{aligned}$$

and similarly $F_P(v) = u$. Also, for $x \in \mathbb{R}^3$ we have

$$F_P(x) = x \iff x - 2\,(x \cdot w)\, w = x \iff 2\,(x \cdot w)\, w = 0 \iff x \cdot w = 0 \iff x \in P.$$

2.36 Theorem: *(Congruent Triangles and Isometries) Given two ordered triangles $[u,v,w]$ and $[u',v',w']$ with $a = a'$, $b = b'$ and $c = c'$, there exists a unique isometry $F : \mathbb{S}^2 \to \mathbb{S}^2$ with $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$.*

Proof: First we note that if such an isometry exists then it is unique because $\{u,v,w\}$ is a basis for $\mathbb{R}^3$. We now construct the required isometry as a composite of reflections. If $u = u'$ then let $F_1 = I$ and if $u \neq u'$ the let F_1 be the orthogonal reflection in the perpendicular bisector of u and u' so that we have $F_1(u) = u'$. Let $u_1 = F_1(u) = u'$, $v_1 = F_1(v)$ and $w_1 = F_1(w)$. Note that if a_1, b_1 and c_1 are the edge lengths of triangle $[u_1, v_1, w_1]$ then we have $a_1 = a$, $b_1 = b$ and $c_1 = c$ since F_1 is an isometry. If $v_1 = v'$ then let $F_2 = I$ and if $v_1 \neq v'$ then let F_2 be the orthogonal reflection in the perpendicular bisector of v_1 and v' so that we have $F_2(v_1) = v'$. Note that in the case that $v_1 \neq v'$, the point $u_1 = u'$ lies on the perpendicular bisector of v_1 and v' because $|v_1 - u_1| = b = |v' - u'| = |v' - u_1|$, and so we have $F_2(u_1) = u_1 = u'$. Let $u_2 = F_2(u_1) = u'$, $v_2 = F_2(v_1) = v'$ and $w_2 = F_2(w_1)$. Note that if a_2, b_2 and c_2 are the edge lengths of triangle $[u_2, v_2, w_2]$ then we have $a_2 = a$, $b_2 = b$ and $c_2 = c$. If $w_2 = w'$ then let $F_3 = I$ and if $w_2 \neq w'$ then let F_3 be the orthogonal reflection in the perpendicular bisector of w_2 and w' so that $F_3(w_2) = w'$. As above, note that $F_3(u_2) = u_2 = u'$ and $F_3(v_2) = v_2 = v'$. Thus the isometry $F = F_3 F_2 F_1$ satisfies $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$.

2.37 Definition: A **rotary reflection** on $\mathbb{S}^2$ is an isometry on $\mathbb{S}^2$ of the form $F_u R_{u,\theta}$ for some $u \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$. A **rotary inversion** on $\mathbb{S}^2$ is an isometry on $\mathbb{S}^2$ of the form $N\,R_{u,\theta}$ for some $u \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$.

2.38 Theorem: *Let $u \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$. Then*
(1) $N\,R_{u,\theta} = -R_{u,\theta} = R_{u,\theta}N$, and
(2) $F_u R_{u,\theta} = R_{u,\theta}F_u = -R_{u,\theta+\pi}$.

Proof: Part 1 holds because $N\,R_{u,\theta} = (-I)R_{u,\theta} = -R_{u,\theta} = R_{u,\theta}(-I) = R_{u,\theta}N$. To prove Part 2, choose $v, w \in \mathbb{S}^2$ so that $\{u, v, w\}$ is a positively oriented orthonormal basis for $\mathbb{R}^3$. Then we have

$$\begin{aligned} F_u R_{u,\theta}(u) &= F_u(u) = -u, \\ F_u R_{u,\theta}(v) &= F_u\big((\cos\theta)\,v + (\sin\theta)\,w\big) = (\cos\theta)\,v + (\sin\theta)\,w, \\ F_u R_{u,\theta}(w) &= F_u\big(-(\sin\theta)\,v + (\cos\theta)\,w\big) = -(\sin\theta)\,v + (\cos\theta)\,w, \end{aligned}$$

and we have

$$\begin{aligned} R_{u,\theta}F_u(u) &= R_{u,\theta}(-u) = -u, \\ R_{u,\theta}F_u(v) &= R_{u,\theta}(v) = (\cos\theta)\,v + (\sin\theta)\,w, \\ R_{u,\theta}F_u(w) &= R_{u,\theta}(w) = -(\sin\theta)\,v + (\cos\theta)\,w, \end{aligned}$$

and we have

$$\begin{aligned} -R_{u,\theta+\pi}(u) &= -u, \\ -R_{u,\theta+\pi}(v) &= -\big(\cos(\theta+\pi)\,v + \sin(\theta+\pi)\,w\big) = (\cos\theta)\,v + (\sin\theta)\,w, \\ -R_{u,\theta+\pi}(w) &= -\big(-\sin(\theta+\pi)\,v + \cos(\theta+\pi)\,w\big) = -(\sin\theta)\,v + (\cos\theta)\,w. \end{aligned}$$

2.39 Theorem: *(The Geometric Classification of Isometries) Every isometry on $\mathbb{S}^2$ is either a rotation, a reflection, or a rotary inversion.*

Proof: Let S be an isometry on $\mathbb{S}^2$. By Theorem 2.36, S is the unique isometry which sends the standard basis vectors e_1, e_2 and e_3 to the points $S(e_1)$, $S(e_2)$ and $S(e_3)$, and the proof of that theorem shows that S is of the form $S = F_3F_2F_1$ where each F_k is either the identity or a reflection. Thus either S is the identity, or S is a reflection, or S is the product of two reflections, or S is the product of three reflections. By Theorem 2.33, the product of two reflections is a rotation (or the identity, when the two reflections are equal), so it suffices to consider the product of three reflections. Suppose that $S = F_wF_vF_u$ where $u, v, w \in \mathbb{S}^2$. If $u = \pm v$ then $F_vF_u = I$ so that $S = F_w$, which is a reflection. Suppose that $u \neq \pm v$, so we have $L_u \cap L_v = \{\pm p\}$ where $p = \frac{u\times v}{|u\times v|}$. By Theorem 2.33, $F_vF_u = R_{p,\theta}$ where we have $u, v \in T_p$ and $\theta = 2\theta_o(u, v)$, and so $S = F_wF_vF_u = F_wR_{p,\theta}$. If $w = \pm p$ then $S = F_wR_{p,\theta} = F_pR_{p,\theta}$, which is a rotary reflection. Suppose $w \neq \pm p$. Let $w' = w$, let $v' = \frac{p\times w}{|p\times w|} \in T_p$ (so that $L_{v'}$ is the unique line through p which is perpendicular to L_w), and let u' be the point in T_p such that $\theta = 2\theta_o(u', v')$ so that we have $R_{p,\theta} = F_{v'}F_{u'}$. Then $S = F_wR_{p,\theta} = F_{w'}F_{v'}F_{u'} = R_{q,\pi}F_{u'}$ where $q = \frac{v'\times w'}{|v'\times w'|}$ (so that $L_{v'} \cap L_{w'} = \{\pm q\}$). If $u' = \pm q$ then $S = R_{q,\pi}F_{u'} = R_{q,\pi}F_q$, which is a rotary reflection. Suppose that $u' \neq \pm q$. Let $u'' = u'$, let $w'' = \frac{q\times u'}{|q\times u'|}$ (so that $L_{w''}$ is the line through q perpendicular to $L_{u''}$) and let v'' be the vector in T_q such that $\pi = 2\theta_o(v'', w'')$ so that we have $R_{q,\pi} = F_{w''}F_{v''}$. Then $S = R_{q,\pi}F_{u'} = F_{w''}F_{v''}F_{u''}$. Since w'' is perpendicular to both u'' and v'' so that $u'', v'' \in T_{w''}$, we have $F_{v''}F_{u''} = R_{w'',\phi}$ where $\phi = 2\theta_o(u'', v'')$, and $S = F_{w''}R_{w'',\phi}$, which is a rotary reflection. In all cases, either S is the identity, or S is a reflection, or S is a rotary reflection, and every rotary reflection is a rotary inversion, by Theorem 2.38.

Projections

2.40 Definition: Let H be the upper hemisphere $H=\{(x,y,z)\in\mathbb{S}^2\,\big|\,z\geq 0\}$ and let D be the disc $D=\{(u,v)\in\mathbb{R}^2\,\big|\,u^2+v^2\leq 1\}$. The **orthogonal projection** from H to D is the map $\phi:H\to D$ given by

$$(u,v)=\phi(x,y,z)=(x,y).$$

Note that this map is invertible and its inverse is the map $\psi:D\to H$ given by

$$(x,y,z)=\psi(u,v)=\big(u,v,\sqrt{1-u^2-v^2}\big).$$

2.41 Note: A curve in H given by $(x,y,z)=\alpha(t)$ for $a\leq t\leq b$ is mapped by the orthogonal projection ϕ to the curve in D given by $(u,v)=\beta(t)=\phi\big(\alpha(t)\big)$. If we are given a formula for $\beta(t)$, say $\beta(t)=\big(u(t),v(t)\big)$, then we can calculate the length L of the curve $(x,y,z)=\alpha(t)=\psi\big(\beta(t)\big)=\Big(u(t),v(t),\sqrt{1-u(t)^2-v(t)^2}\Big)$ using the formula

$$L=\int_{t=a}^{b}\big|\alpha'(t)\big|\,dt.$$

When $\beta(t)$ is given in polar coordinates by $\beta(t)=\big(r(t)\cos\theta(t),r(t)\sin\theta(t)\big)$, we can calculate the length L of the curve $(x,y,z)=\alpha(t)$ as follows.

$$\begin{aligned}
\alpha(t)&=\Big(r\cos\theta\,,\,r\sin\theta\,,\,\sqrt{1-r^2}\Big)\\
\alpha'(t)&=\Big(r'\cos\theta-r\sin\theta\cdot\theta'\,,\,r'\sin\theta+r\cos\theta\,\theta'\,,\,\tfrac{-r}{\sqrt{1-r^2}}\,r'\Big)\\
L&=\int_{t=a}^{b}\sqrt{(r'\cos\theta-r\sin\theta\,\theta')^2+(r'\sin\theta+r\cos\theta\,\theta')^2+\big(\tfrac{r}{\sqrt{1-r^2}}\,r'\big)^2}\,dt\\
&=\int_{t=a}^{b}\sqrt{(r')^2+r^2(\theta')^2+\tfrac{r^2}{1-r^2}(r')^2}\,\,dt\\
&=\int_{t=a}^{b}\sqrt{\tfrac{1}{1-r^2}(r')^2+r^2(\theta')^2}\,\,dt.
\end{aligned}$$

2.42 Note: When $R\subseteq\{(r,\theta)\,\big|\,0\leq r\leq 1\,,\,0\leq\theta\leq 2\pi\}$ and $\sigma:R\to H$, by using some vector calculus, one can show that the region in H which is given parametrically by $(x,y,z)=\sigma(r,\theta)=\Big(r\cos\theta\,,\,r\sin\theta\,,\,\sqrt{1-r^2}\Big)$ has area

$$\begin{aligned}
A&=\iint_R\big|\sigma_r\times\sigma_\theta\big|\,dr\,d\theta\\
&=\iint_R\Big|\big(\cos\theta\,,\,\sin\theta\,,\,\tfrac{-r}{\sqrt{1-r^2}}\big)\times\big(-r\sin\theta\,,\,r\cos\theta\,,\,0\big)\Big|\,dr\,d\theta\\
&=\iint_R\big(\tfrac{r^2\cos\theta}{\sqrt{1-r^2}}\,,\,\tfrac{r^2\sin\theta}{\sqrt{1-r^2}}\,,\,r\big)\Big|\,dr\,d\theta=\iint_R\sqrt{\tfrac{r^4}{1-r^2}+r^2}\,\,dr\,d\theta\\
&=\iint_R\tfrac{r}{\sqrt{1-r^2}}\,dr\,d\theta.
\end{aligned}$$

2.43 Remark: We can project orthogonally onto any plane through the origin. When $u\in\mathbb{S}^2$, $H=D(u,\frac{\pi}{2})=\{x\in\mathbb{S}^2\,\big|\,x\cdot u\geq 0\}$ and $D=\{x\in\mathbb{R}^3\,\big|\,x\cdot u=0,|x|\leq 1\}$, the **orthogonal projection** $\phi:H\to D$ and its inverse $\psi:D\to H$ are given by

$$y=\phi(x)=x-(x\cdot u)u\quad\text{and}\quad x=\psi(y)=y+\sqrt{1-|y|^2}\,u\,.$$

2.44 Definition: Let $S = \mathbb{S}^2 \setminus \{\pm(0,0,1)\}$ and $R = \big\{(\theta, z)\big|0 \le \theta < 2\pi, -1 \le z \le 1\big\}$. The **Lambert cylindrical equal-area projection** is the map from S to R which is obtained by first projecting radially outwards from the z-axis to the cylinder $x^2 + y^2 = 1$ and then cutting the cylinder along the line $x = 1$, $y = 0$ and unrolling it into the rectangle R. The map $\phi : S \to R$ is given by

$$\phi\Big(\sqrt{1-z^2}\cos\theta\,,\ \sqrt{1-z^2}\sin\theta\,,\ z\Big) = (\theta, z)$$

and its inverse $\psi : R \to S$ is given by

$$\psi(\theta, z) = \Big(\sqrt{1-z^2}\cos\theta\,,\ \sqrt{1-z^2}\sin\theta\,,\ z\Big).$$

2.45 Theorem: *The Lambert cylindrical equal area projection preserves area.*

Proof: This can be seen to be a consequence of Theorem 2.4 (The Spherical Area Theorem), but we shall provide a proof which uses some vector calculus. The area A of a region $D \subseteq R = \big\{(r, \theta)\big|0 \le \theta < 2\pi, -1 \le z \le 1\big\}$ is given by

$$A = \iint_D 1\, d\theta\, dz.$$

The area of its inverse mage under ϕ is the area of its image under ψ which is

$$\begin{aligned} B &= \iint_D \big|\psi_\theta \times \psi_z\big|\, d\theta\, dz \\ &= \iint_D \left|\Big(-\sqrt{1-z^2}\sin\theta\,,\ \sqrt{1-z^2}\cos\theta\,,\ 0\Big) \times \Big(\tfrac{-z}{\sqrt{1-z^2}}\cos\theta\,,\ \tfrac{-z}{\sqrt{1-z^2}}\sin\theta\,,\ 1\Big)\right| d\theta\, dz \\ &= \iint_D \Big|\big(\sqrt{1-z^2}\cos\theta\,,\ \sqrt{1-z^2}\sin\theta\,,\ z\big)\Big|\, d\theta\, dz \\ &= \iint_D \sqrt{(1-z^2)+z^2}\, d\theta\, dz = \iint_D 1\, d\theta\, dz = A. \end{aligned}$$

2.46 Remark: We can obtain an alternate equal-area projection by composing ϕ with a map that scales the rectangle R by a scaling factor of c in the θ direction and by $\frac{1}{c}$ in the z-direction. We can also choose to project radially outwards from any line through the origin to the cylinder centred along that line (we can choose a line other than the z-axis).

2.47 Definition: Let H be the open hemisphere $H = \big\{(x, y, z) \in \mathbb{S}^2\big|z > 0\big\}$. The **gnomic projection** (or the **gnomonic projection**) from H to $\mathbb{R}^2$ is the map $\phi : H \to \mathbb{R}^2$ obtained by projecting radially outwards from the origin to the plane $z = 1$ which we identify with $\mathbb{R}^2$. The line through $(0,0,0)$ and (x, y, z) meets the plane $z = 1$ at the point $\big(\frac{x}{z}, \frac{y}{z}, 1\big)$, and so the map ϕ is given by

$$(u, v) = \phi(x, y, z) = \Big(\frac{x}{z}\,,\ \frac{y}{z}\Big).$$

Its inverse is the map $\psi : \mathbb{R}^2 \to H$ given by

$$(x, y, z) = \psi(u, v) = \frac{(u, v, 1)}{\big|(u, v, 1)\big|} = \Big(\frac{u}{\sqrt{u^2+v^2+1}}\,,\ \frac{v}{\sqrt{u^2+v^2+1}}\,,\ \frac{1}{\sqrt{u^2+v^2+1}}\Big).$$

2.48 Remark: We could, if we wanted, also define gnomic projections from other open hemispheres, for example from the hemisphere $\big\{(x, y, z) \in \mathbb{S}^2 \,\big|\, x > 0\big\}$ or from the hemisphere $\big\{(x, y, z) \in \mathbb{S}^2 \,\big|\, x < 0\big\}$.

2.49 Theorem: *The gnomic projection maps great circles on $\mathbb{S}^2$, intersected with the open upper hemisphere H, to lines in $\mathbb{R}^2$.*

Proof: When L is a line in $\mathbb{S}^2$ with pole not equal to $\pm(0,0,1)$, and P is the plane in $\mathbb{R}^3$ which contains L so that $L = \mathbb{S}^2 \cap P$, and M is the intersection of P with the plane $z = 1$, it is easy to see with the help of a picture that ϕ maps $L \cap H$ to M. Here is an analytic proof. Let $L = \mathbb{S}^2 \cap P$ where P is the plane $ax + by + cz = 0$ with $c \neq 0$. The line of intersection of P with the plane $z = 1$ is the line given by $ax + by + c = 0$, $z = 1$. We show that ϕ maps $L \cap H$ to the line M in the uv-plane with equation $au + bv + c = 0$. Let $(x, y, z) \in L \cap H$. Then we have $ax + by + cz = 0$ and $z > 0$. For $(u, v) = \phi(x, y, z)$ we have

$$au + bv + c = a\,\tfrac{x}{z} + b\,\tfrac{y}{z} + c = \tfrac{ax+by+cz}{z} = \tfrac{0}{z} = 0$$

and so the point (u, v) lies on the line M. Conversely, let $(u, v) \in M$ so that $au+bv+c = 0$ and let $(x, y, z) = \psi(u, v)$. Then $z = \frac{1}{u^2+v^2+1} > 0$ so that $(x, y, z) \in H$ and we have

$$ax + by + cz = \tfrac{au}{u^2+v^2+1} + \tfrac{bv}{u^2+v^2+1} + \tfrac{c}{u^2+v^2+1} = \tfrac{au+bv+c}{u^2+v^2+1} = 0$$

so the point $(x, y, z) \in L$.

2.50 Definition: Let $S = \mathbb{S}^2 \setminus \{(0,0,1)\}$. The **stereographic projection** from S to $\mathbb{R}^2$ is the map $\phi : S \to \mathbb{R}^2$ which sends the point (x, y, z) on the sphere to the point of intersection (u, v) of the line through $(0,0,1)$ and (x, y, z) with the plane $z = 0$. Given $(x, y, z) \in S$, the line through $(0,0,1)$ and (x, y, z) is given by

$$(u, v, w) = \alpha(t) = (0,0,1) + t(x, y, z-1) = \big(tx\,,\, ty\,,\, 1 + t(z-1)\big) \text{ for } t \in \mathbb{R}.$$

The point of intersection of this line with the plane $z{=}0$ occurs when $w{=}1{+}t(z{-}1) = 0$, that is when $t = 1/(1-z)$. The point of intersection is $\alpha\big(\frac{1}{1-z}\big) = \big(\frac{x}{1-z}, \frac{y}{1-z}, 0\big)$, so the map ϕ is given by

$$(u, v) = \phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Given $(u, v) \in \mathbb{R}^2$, the line through $(0,0,1)$ and (u, v) is given by

$$(x, y, z) = \beta(t) = (0,0,1) + t(u, v, -1) = (tu, tv, 1-t) \text{ for } t \in \mathbb{R}.$$

The point of intersection with the unit sphere occurs when $|\beta(t)| = 1$, that is when we have $(tu)^2 + (tv)^2 + (1-t)^2 = 1$, that is $t^2u^2 + t^2v^2 - 2t + t^2 = 0$, or $t\,(tu^2 + tv^2 + t - 2) = 0$. The intersection occurs when $t = \frac{2}{u^2+v^2+1}$, and so the inverse $\psi : \mathbb{R}^2 \to S$ is given by

$$(x, y, z) = \psi(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

2.51 Note: Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$ and let $w \in \mathbb{R}^n$. Choose a differentiable map $\alpha : A \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^n$ (where A is an open interval in $\mathbb{R}$ with $0 \in A$) with $\alpha(0) = a$ and $\alpha'(0) = w$ and let $\beta(t) = f\big(\alpha(t)\big)$. By the Chain Rule we have $\beta'(t) = Df\big(\alpha(t)\big)\alpha'(t)$ so in particular $\beta'(0) = Df\big(\alpha(0)\big)\alpha'(0) = Df(a)w$. Thus the map f sends the vector w to the vector $Df(a)w$.

2.52 Definition: Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in U$ and let $c > 0$. We say that f is a **local scaling** near the point a of **scaling factor** c when the columns of $Df(a)$ are orthogonal and of length c, in other words when $Df(a)^T Df(a) = c^2 I$. We say that f is **conformal** at a when f preserves the angles between vectors at a, that is when

$$\frac{\big(Df(a)w_1\big) \cdot \big(Df(a)w_2\big)}{\big|Df(a)w_1\big|\big|Df(a)w_2\big|} = \frac{w_1 \cdot w_2}{|w_1||w_2|} \text{ for all vectors } 0 \neq w_1, w_2 \in \mathbb{R}^n.$$

2.53 Note: When $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a local scaling near $a \in U$ of scaling factor $c > 0$, f is conformal at a because for $0 \neq w_1, w_2 \in \mathbb{R}^n$ we have

$$\big(Df(a)w_1\big) \bullet \big(Df(a)w_2\big) = \big(Df(a)w_2\big)^T \big(Df(a)w_1\big) = w_2^T (c^2 I) w_1 = c^2 w_1 \bullet w_2$$

hence $\big|Df(a)w_i\big|^2 = \big(Df(a)w_i\big) \bullet \big(Df(a)w_i\big) = c^2 |w_i|^2$ so that $\big|Df(a)w_i\big| = c|w_i|$ for $i = 1, 2$, and so

$$\frac{\big(Df(a)w_1\big) \bullet \big(Df(a)w_2\big)}{\big|Df(a)w_1\big|\big|Df(a)w_2\big|} = \frac{c^2\, w_1 \bullet w_2}{c|w_1| \cdot c|w_2|} = \frac{w_1 \bullet w_2}{|w_1||w_2|}.$$

2.54 Theorem: *The inverse stereographic projection map is a local scaling near (u, v) of scaling factor $c = \frac{2}{u^2+v^2+1}$.*

Proof: We have

$$D\psi = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} = \frac{2}{(u^2+v^2+1)^2} \begin{pmatrix} -u^2+v^2+1 & -2uv \\ -2uv & u^2-v^2+1 \\ 2u & 2v \end{pmatrix}$$

and a quick calculation yields

$$(D\psi)^T (D\psi) = \frac{4}{(u^2+v^2+1)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.55 Theorem: *The stereographic projection maps circles through $(0, 0, 1)$ in $\mathbb{S}^2$, with the point $(0, 0, 1)$ removed, to lines in $\mathbb{R}^2$, and it sends circles not through $(0, 0, 1)$ in $\mathbb{S}^2$ to circles in $\mathbb{R}^2$.*

Proof: Let $n = (0, 0, 1)$. We leave the first part of the theorem as an exercise, and we show that the image under ϕ of each circle in $\mathbb{S}^2 \setminus \{n\}$ is a circle in $\mathbb{R}^2$. Let C be a circle on $\mathbb{S}^2 \setminus \{n\}$. Say C is the intersection of $\mathbb{S}^2$ with the plane P given by $ax + by + cz + d = 0$. Since $n \notin C$ we have $n \notin P$ and so $c + d \neq 0$. Since the distance from P to the origin is less than 1 we have $\text{dist}(0, P)^2 = \frac{d^2}{a^2+b^2+c^2} < 1$. For $(u, v) \in \mathbb{R}^2$ we have

$$\begin{aligned}
(u,v) \in \phi(C) &\iff \psi(u,v) \in P \iff \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2-1}\Big) \in P \\
&\iff a \cdot \frac{2u}{u^2+v^2+1} + b \cdot \frac{2v}{u^2+v^+1} + c \cdot \frac{u^2+v^2-1}{u^2+v^2+1} + d = 0 \\
&\iff 2a\, u + 2b\, v + c(u^2+v^2-1) + d(u^2+v^2+1) = 0 \\
&\iff (c+d)u^2 + 2a\, u + (c+d)v^2 + 2b\, v = c - d \\
&\iff u^2 + \tfrac{2a}{c+d}\, u + v^2 + \tfrac{2b}{c+d}\, v = \tfrac{c-d}{c+d} \\
&\iff \big(u + \tfrac{a}{c+d}\big)^2 + \big(v + \tfrac{b}{c+d}\big)^2 = \tfrac{c-d}{c+d} + \tfrac{a^2}{(c+d)^2} + \tfrac{b^2}{(c+d)^2} = \tfrac{a^2+b^2+c^2-d^2}{(c+d)^2} \\
&\iff (u,v) \text{ lies on the circle centred at } \big(\tfrac{-a}{c+d}, \tfrac{-b}{c+d}\big) \text{ of radius } r = \tfrac{\sqrt{a^2+b^2+c^2-d^2}}{|c+d|}.
\end{aligned}$$

2.56 Remark: We defined the stereographic projection from $\mathbb{S}^2 \setminus \{n\}$ to $\mathbb{R}^2$ where n is the north pole $n = (0, 0, 1)$, but we could, if we wanted, also define the stereographic projection from $\mathbb{S}^2 \setminus \{u\}$ to $\mathbb{R}^2$ for any point $u \in \mathbb{S}^2$.