# MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 9

**1:** (a) Let $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 4 & 1 \\ 5 & 1 & 3 \end{pmatrix} \in M_3(\mathbb{Z}_7)$. Find an invertible matrix $P \in M_3(\mathbb{Z}_7)$ such that $P^T A\, P$ is diagonal.

Solution: We follow the procedure described in Theorem 9.12 in the Lecture Notes, using column and row operations to put $A$ into diagonal form. At each stage we indicate the operations used and give the elementary matrix for the column operations.

$$C_2 \mapsto C_2 + 3C_1 \quad \begin{pmatrix} 2 & 0 & 5 \\ 1 & 0 & 1 \\ 5 & 2 & 3 \end{pmatrix} \quad R_2 \mapsto R_2 + 3R_1 \quad \begin{pmatrix} 2 & 0 & 5 \\ 0 & 0 & 2 \\ 5 & 2 & 3 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + C_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 5 & 2 & 1 \end{pmatrix} \quad R_3 \mapsto R_3 + R_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_2 \mapsto C_2 + C_3 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & 1 \end{pmatrix} \quad R_2 \mapsto R_2 + R_3 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 3 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + 5C_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 3 & 2 \end{pmatrix} \quad R_3 \mapsto R_3 + 5R_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we can take

$$\begin{aligned} P = E_1E_2E_3E_4 &= \left( \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 5 \\ 0 & 1 & 6 \end{pmatrix}. \end{aligned}$$

(b) Show that in $M_3(\mathbb{Z}_7)$ we have $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \not\cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For $P = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $P^TAP = E$ so that $A \cong E$. Suppose, for a contradiction, that there exists an invertible matrix $Q \in M_3(\mathbb{Z}_7)$ such that $Q^TBQ = E$. Write $Q$ in block form as $Q = \begin{pmatrix} R & x \\ y^T & z \end{pmatrix}$ where $R \in M_2(\mathbb{Z}_7)$, $x, y \in {\mathbb{Z}_7}^2$ and $z \in \mathbb{Z}_7$. Since $Q^TBQ = E$, we can equate the upper left $2 \times 2$ block on both sides to get $R^T \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Take the determinant on both sides to get $3\det(R)^2 = 1$ so that $\det(R)^2 = \frac{1}{3} = 5$. This is not possible since 5 is not a square in $\mathbb{Z}_7$ $\big($indeed $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$ and $(\pm 3)^2 = 2\big)$.

**2:** (a) Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 4 \end{pmatrix} \in M_3(\mathbb{Z}_5)$. Find a matrix $P \in M_3(\mathbb{Z}_5)$ such that $P^T A P = I$.

Solution: We use column and row operations to put $A$ into diagonal form. At each stage we indicate the operations used and give the elementary matrix for the column operations.

$$C_2 \mapsto C_2 + 2C_1 \quad \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 4 \end{pmatrix} \quad R_2 \mapsto R_2 + 2R_1 \quad \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 3 \\ 3 & 3 & 4 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + C_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 3 & 3 & 2 \end{pmatrix} \quad R_3 \mapsto R_3 + R_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 2 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + 2C_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 3 \end{pmatrix} \quad R_3 \mapsto R_3 + 2R_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

This shows that if we let

$$Q = E_1 E_2 E_3 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

then we have $Q^T A Q = \text{diag}(2, 1, 3)$. Also note that

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so if we let

$$P = Q \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

then we have $P^T A P = I$.

(b) Let $A \in M_n(\mathbb{Z}_3)$ with $A^T = A$ and $\det A = 1$. Show that there exists $P \in M_n(\mathbb{Z}_3)$ such that $P^T A P = I$.

Solution: We know that $A$ is congruent to a diagonal matrix $D = \text{diag}(d_1, \cdots, d_n)$. Since $A$ is invertible, all of the entries $d_i$ are non-zero so we have $d_i \in \{1, 2\}$ for all $i$. Using the operations $R_i \leftrightarrow R_j$ and $C_i \leftrightarrow C_j$, we can rearrange the entries $d_i$ of $D$, so the matrix $A$ is congruent to a matrix of the form

$$E = \begin{pmatrix} I_k & \\ & 2I_{n-k} \end{pmatrix}$$

for some $k$ with $0 \le k \le n$. Notice that

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $2I_2$ is congruent to $I_2$. It follows that, up to congruence, we can replace copies of the $2 \times 2$ block $2I_2$ in the above matrix $E$ by copies of $I_2$, and hence $A$ is congruent either to $I_n$ or to the matrix $\begin{pmatrix} I_{n-1} & 0 \\ 0 & 2 \end{pmatrix}$. But $A$ cannot be congruent to the latter matrix because for an invertible matrix $P$ we have $\det P \in \{1, 2\}$ so that $(\det P)^2 = 1$ and so $\det(P^T A P) = (\det P)^2 \det A = \det A = 1$. Thus $A$ is congruent to $I = I_n$.