MATH 245 Linear Algebra 2, Exercises for Chapter 8

1: (a) Let $A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & -3 & 2 \\ 4 & 2 & 0 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

(b) Let $A = \begin{pmatrix} 5+i & -6 \\ 2 & -2+i \end{pmatrix}$. Find a unitary matrix P and an upper-triangular matrix T so that $P^*AP = T$.

2: For $0 \neq u \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$, let $R_{u,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the rotation about the vector u by the angle θ (where the direction of rotation is determined by the right-hand rule: the right thumb points in the direction of u and the fingers curl in the direction of rotation).

(a) Let $u = (1, 1, -1)^T$ and let $\theta = \frac{\pi}{3}$. Find $A = \left[R_{u,\theta}\right]_{\mathcal{S}}$ where $\mathcal{S}$ is the standard basis for $\mathbb{R}^3$.

(b) Let $B = \begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix}$. Find $c > 0$, $0 \neq u \in \mathbb{R}^3$ and $0 \leq \theta \leq \pi$ such that $B = \left[c\, R_{u,\theta}\right]$.

3: Find a singular value decomposition $Q^*AP = S$ for the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_{4\times 2}(\mathbb{R})$.

4: Let $A \in M_n(\mathbb{C})$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A (listed with repetition according to algebraic multiplicity). Show that the following are equivalent.

1. $A\,A^* = A^*A$.
2. $A^* = f(A)$ for some polynomial f.
3. $A^* = AP$ for some unitary matrix P.
4. $\sum_{i,j} \left|A_{i,j}\right|^2 = \sum_i |\lambda_i|^2$.

5: A matrix $A \in M_{n\times n}(\mathbb{C})$ is called Hermitian positive-definite when $A^* = A$ and the eigenvalues of A are all positive. Let $H_n(\mathbb{C})$ denote the set of Hermitian positive-definite matrices in $M_{n\times n}(\mathbb{C})$.

(a) Let $A \in H_n(\mathbb{C})$. Show that if $Q^*AP = S$ is a singular value decomposition of A, then $Q = P$.

(b) Show that every element of $H_n(\mathbb{C})$ has a unique square root in $H_n(\mathbb{C})$.

(c) Show that for every $A \in GL_n(\mathbb{C})$ there exist unique $R \in H_n(\mathbb{C})$ and $\Theta \in U_n(\mathbb{C})$ such that $A = R\Theta$.

6: Let U be a finite-dimensional inner product space over $\mathbb{R}$ and let $L : U \to U$ be linear. Suppose $L^*L = L\,L^*$. Show that there is an orthonormal basis $\mathcal{U}$ for U such that $[L]_{\mathcal{U}}$ is in the block-diagonal form

$$[L]_{\mathcal{U}} = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_k & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_l & b_l \\ & & & & & & -b_l & a_l \end{pmatrix}$$

where each 1×1 block corresponds to a real eigenvalue λ_j of L, and each 2×2 block corresponds to a pair of conjugate complex eigenvalues $a_j \pm i\, b_j$.