MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 7

1: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $W = \mathbb{F}^\infty$ with its standard inner product. Let $U = \left\{ a = (a_1, a_2, \cdots) \in W \,\middle|\, \sum_{k=1}^{\infty} a_k = 0 \right\}$. Let $\mathcal{S} = \{e_1, e_2, e_3, \cdots\}$ be the standard basis for W, where $e_n = (e_{n,1}, e_{n,2}, e_{n,3}, \cdots)$ with $e_{n,k} = \delta_{n,k}$.

(a) Recall that the *annihilator* of U in W^* is the vector space $U^0 = \{ f \in W^* | f(a) = 0 \text{ for all } a \in U \}$. Show that $\dim(U^0) = 1$.

Solution: Define $f : V \to \mathbb{F}$ by $f(a) = \sum_{k=1}^{\infty} a_k$. Note that f is well-defined since only finitely many of the terms a_k are non-zero, and f is linear, so we have $f \in V^*$. We claim that $U^0 = \text{Span}\{f\}$. For $a = (a_1, a_2, \cdots) \in U$, we have $f(a) = \sum_{k=1}^{\infty} a_k = 0$, so $f \in U^0$ and hence $\text{Span}\{f\} \subset U^0$. Conversely, let $g \in U^0$ so that $g(u) = 0$ for all $u \in U$. Notice that for all $k = 1, 2, 3, \cdots$ we have $e_1 - e_k \in U$, so $0 = g(e_1 - e_k) = g(e_1) - g(e_k)$, and hence $g(e_k) = g(e_1)$. For all $a \in V$, we have

$$g(a) = g\big(\sum_{k=1}^{\infty} a_k e_k \big) = \sum_{k=1}^{\infty} a_k \, g(e_k) = \sum_{k=1}^{\infty} a_k \, g(e_1) = \big(\sum_{k=1}^{\infty} a_k \big) \, g(e_1) = g(e_1) \, f(a) \, .$$

Thus $g = g(e_1) \, f \in \text{Span}\{f\}$ and hence $U^0 \subset \text{Span}\{f\}$.

(b) Let $\mathcal{F} = \{f_1, f_2, f_3, \cdots\}$ where $f_n \in W^*$ is determined by $f_n(e_k) = \delta_{n,k}$. Show that $\mathcal{F}$ is linearly independent but does not span W^*.

Solution: We claim that $\mathcal{F}$ is linearly independent. Suppose that some (finite) linear combination of the elements of $\mathcal{F}$ is equal to zero, say $\sum_{i=1}^{n} c_i \, f_i = 0$. Then for every $a \in V$ we have $\sum_{i=1}^{n} c_i \, f_i(a) = 0$, and in particular for every $k = 1, 2, 3, \cdots$ we have $0 = \sum_{i=1}^{n} c_i \, f_i(e_k) = \sum_{i=1}^{n} c_i \, \delta_{i,k} = c_k$. Thus $\mathcal{F}$ is linearly independent.

On the other hand, we claim that $\mathcal{F}$ does not span V^*. Let $f \in V^*$ be the map from part (b) given by $f(a) = \sum_{k=1}^{\infty} a_k$. Notice that f cannot be equal to any (finite) linear combination of the elements of $\mathcal{F}$, since for $g = \sum_{i=1}^{n} c_i \, f_i$ we have $g(e_{n+1}) = 0$ while $f(e_{n+1}) = 1$, so $g \neq f$. Thus $\mathcal{F}$ does not span V^*.

(c) Define $E : W \to W^{**}$ by $E(a)(f) = f(a)$, where $a \in W$ and $f \in W^*$. Show that E is 1:1 but not onto.

Solution: Note that E is linear, so to show that E is 1:1 it suffices to show that $\text{Null}(E) = \{0\}$. Let $a \in \text{Null}(E)$ so $E(a) = 0$. Then for all $f \in V^*$ we have $f(a) = E(a)(f) = 0$. In particular, for all $k = 1, 2, 3 \cdots$ we have

$$0 = f_k(a) = f_k\big(\sum_{i=1}^{\infty} a_i \, e_i \big) = \sum_{i=1}^{\infty} f_k(e_i) = \sum_{i=1}^{\infty} a_i \, \delta_{k,i} = a_k$$

and so $a = 0$.

We claim that E is not onto. Extend the linearly independent set $\mathcal{F}$ to a basis $\mathcal{F} \cup \mathcal{G}$ for V^* (where $\mathcal{F}$ and $\mathcal{G}$ are disjoint). Let $h : V^* \to \mathbb{F}$ be the (unique) linear map given by $h(f_k) = 1$ for all $k = 1, 2, 3, \cdots$ and $h(g) = 0$ for all $g \in \mathcal{G}$. Notice that h cannot be in the range of E since given $a = (a_1, a_2, \cdots) \in V$ we can choose k so that $a_k = 0$, and then we have $E(a)(f_k) = f_k(a) = 0$ while $h(f_k) = 1$, so $E(a) \neq h$.

(d) Define $L : W \to W$ by $L(a)_k = \sum_{i=k}^{\infty} a_i$, where $a \in W$. Show that L has no adjoint.

Solution: Notice that for all $k = 1, 2, 3, \cdots$ we have $L(e_k) = (1, 1, \cdots, 1, 0, 0, 0, \cdots) = \sum_{i=1}^{k} e_i$, and so $\langle L(e_k), e_1 \rangle = 1$. Suppose, for a contradiction, that L had an adjoint L^*. Let $a = L^*(e_1) \in V$. Choose k so that $a_k = 0$. Then $\langle e_k, a \rangle = \overline{a_k} = 0$. But this contradicts the fact that $\langle e_k, a \rangle = \langle e_k, L^*(e_1) \rangle = \langle L(e_k), e_1 \rangle = 1$.

2: Let $U=P(\mathbb{R})=\mathbb{R}[x]$. Fix $p\in U$. Let $L:U\to U$ be multiplication by p, that is $L(f)=pf$ for all $f\in U$, and let $D:U\to U$ be the differentiation operator, that is $D(f)=f'$ for all $f\in U$.

(a) Show that if we use the inner product on U given by $\left\langle \sum a_i x^i, \sum b_i x^i\right\rangle = \sum a_i b_i$ then both L and D have adjoints.

Solution: When we use the inner product $\left\langle \sum a_i x^i, \sum b_i x^i\right\rangle = \sum a_i b_i$, the standard basis $\mathcal{S}=\{e_1,e_2,e_3,\cdots\}$ is orthonormal. The differentiation operator is given by

$$D\Big(\sum_{i=0}^{\infty} a_i x^i\Big) = \sum_{i=1}^{\infty} i\, a_i x^{i-1} = \sum_{i=0}^{\infty}(i+1)a_{i+1}x^i\,.$$

Informally, we note that, with respect to the standard basis, we have

$$[D]=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \\ 0 & 0 & 0 & 3 & \\ \vdots & & & & \end{pmatrix}\ ,\ [D]^*=\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & \\ \vdots & & & \end{pmatrix}.$$

Since the (infinite) matrix $[D]^*$ has finitely many non-zero entries in each column, it is the matrix for a well-defined adjoint operator D^*. To be more formal, we define $E:U\to U$ to be the linear map suggested by the above matrix, that is we define E by

$$E\Big(\sum_{i=0}^{\infty} b_i x^i\Big) = \sum_{i=0}^{\infty}(i+1)b_i x^{i+1} = \sum_{i=1}^{\infty} i\ b_{i-1}x^i\,.$$

Then for $a=\sum a_i x^i$ and $b=\sum b_i x^i$ we have

$$\big\langle Da,b\big\rangle = \Big\langle \sum_{i=0}^{\infty}(i+1)a_{i+1}x^i\,,\ \sum_{i=0}^{\infty} b_i x^i\Big\rangle = \sum_{i=0}^{\infty}(i+1)a_{i+1}b_i\ ,\ \text{and}$$
$$\big\langle a,Eb\big\rangle = \Big\langle \sum_{i=0}^{\infty} a_i x^i\,,\ \sum_{i=1}^{\infty} i\, b_{i-1}x^i\Big\rangle = \sum_{i=1}^{\infty} i\, a_i b_{i-1} = \sum_{i=0}^{\infty}(i+1)a_{i+1}b_i = \big\langle Da,b\big\rangle,$$

and so we have $D^*=E$.

If we write the fixed polynomial $p\in U$ as $p(x)=c_0+c_1x+\cdots+c_mx^m$, then the multiplication operator L is given by

$$L\Big(\sum_{i=0}^{\infty} a_i x^i\Big) = (c_0a_0)+(c_1a_0+c_0a_1)x+(c_2a_0+c_1a_1+c_0a_2)x^2+\cdots\,.$$

Informally, we note that, respect to the standard basis, we have

$$[L]=\begin{pmatrix} c_0 & 0 & 0 & \cdots \\ c_1 & c_0 & 0 & \\ \vdots & c_1 & c_0 & \\ c_m & \vdots & c_1 & \\ 0 & c_m & \vdots & \\ 0 & 0 & c_m & \\ \vdots & & & \end{pmatrix}\ ,\ [L]^*=\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_m & 0 & 0 & \cdots \\ 0 & c_0 & c_1 & c_2 & \cdots & c_m & 0 & \\ 0 & 0 & c_0 & c_1 & c_2 & \cdots & c_m & \\ \vdots & & & & & & & \end{pmatrix}.$$

Since the (infinite) matrix $[L]^*$ has finitely many non-zero entries in each column, it can be used to determine a well-defined adjoint for L. Let

$$Q(\mathbb{R}) = \Big\{ \sum_{i=-\infty}^{\infty} a_i x^i \Big| a_i\in\mathbb{R} \text{ with } a_i=0 \text{ for all but finitely many indices } i\Big\}.$$

Note that $\big\{\cdots,x^{-2},x^{-1},1,x,x^2,\cdots\big\}$ is an orthonormal basis for $Q(\mathbb{R})$. The orthogonal projection of $Q(\mathbb{R})$ onto $U=P(\mathbb{R})$ is given by

$$\text{Proj}_U\Big(\sum_{i=-\infty}^{\infty} a_i x^i\Big) = \sum_{i=0}^{\infty} a_i x^i\,.$$

Define $q\in Q(\mathbb{R})$ by $q(x)=p\left(\frac{1}{x}\right)$ so that when $p(x)=c_0+c_1x+\cdots+c_mx^m$ we have

$$q(x)=c_mx^{-m}+c_{m-1}x^{-(m-1)}+\cdots+c_1x^{-1}+c_0\,.$$

Define $M : U \to U$ to be the linear map given by

$$M(g) = \mathrm{Proj}_U(q\,p)$$

We claim that M is the adjoint of L. First, we note that to show that M is the adjoint of L, it suffices to show that $\langle Lx^i, x^j\rangle = \langle x^i, Mx^j\rangle$ for all i, j, because then, for all $a = \sum a_i x^i$ and $b = \sum b_j x^j\rangle$ we have

$$\Big\langle L\big(\sum_{i=0}^{\infty} a_i x^i\big)\,,\ \sum_{j=0}^{\infty} b_j x^j \Big\rangle = \sum_{i,j} a_i b_j \big\langle L(x^i), x^j\big\rangle = \sum_{i,j} a_i b_j \big\langle x^i, M(x^j)\big\rangle = \Big\langle \sum_{i=0}^{\infty} a_i x^i\ ,\ M\big(\sum b_j x^j\big)\Big\rangle\,.$$

Next we note that

$$\begin{aligned}
\big\langle lx^i, x^j\big\rangle &= \big\langle x^i p(x), x^j\big\rangle \\
&= \big\langle c_0 x^i + c_1 x^{i+1} + \cdots + c_m x^{i+m},\ x^j\big\rangle \\
&= \text{the coefficient of } x^j \text{ in } \big(c_0 x^i + c_1 x^{i+1} + \cdots + c_m x^{i+m}\big) \\
&= \begin{cases} c_{j-i} & \text{if } i \le j \le i+m, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\big\langle x^i, Mx^j\big\rangle &= \big\langle x^i\ ,\ \mathrm{Proj}_U x^j q(x)\big\rangle \\
&= \big\langle x^i\ ,\ \mathrm{Proj}_U\big(c_m x^{j-m} + \cdots + c_1 x^{j-1} + c_0 x^j\big)\big\rangle \\
&= \text{the coefficient of } x^i \text{ in } \big(c_m x^{j-m} + \cdots + c_1 x^{j-1} + c_0 x^j\big) \\
&= \begin{cases} c_{j-i} & \text{if } j-m \le i \le j, \\ 0 & \text{otherwise.} \end{cases} \\
&= \big\langle Lx^i, x^j\big\rangle
\end{aligned}$$

Thus $L^* = M$, as claimed.

(b) Show that if we use the inner product given by $\langle f, g\rangle = \int_a^b fg$, then L has an adjoint but D does not. Hint: to show that D does not have an adjoint, you might find it useful to show first that there is no $g \in U$ with the property that $\langle g, f\rangle = f(b) - f(a)$ for every $f \in U$, and then use Integration by Parts.

Solution: Note that for all $f, g \in P(\mathbb{R})$ we have

$$\langle Lf, g\rangle = \langle pf, g\rangle = \int_a^b pfg = \langle f, pg\rangle = \langle f, Lg\rangle$$

and so we see that L^* exists and is equal to L (so L is self-adjoint).

To prove that D has no adjoint, we first follow the hint. Suppose, for a contradiction, that there exists $g \in P(\mathbb{R})$ with the property that $\langle g, f\rangle = f(b) - f(a)$ for all $f \in P(\mathbb{R})$. Choose such a polynomial g. Then, taking $f(x) = (x-a)^2(x-b)^2 g(x)$, so that $f(a) = f(a) = 0$, we obtain

$$0 = \big\langle g(x)\,,\,(x-a)^2(x-b)^2 g(x)\big\rangle = \int_a^b \big((x-a)(x-b)g(x)\big)^2\,dx = \big|(x-a)(x-b)g(x)\big|^2$$

and so $(x-a)(x-b)g(x) = 0 \in P(\mathbb{R})$, and hence $g(x) = 0 \in P(\mathbb{R})$. But clearly $g(x) = 0$ does not have the required property, so no such polynomial g exists.

Now suppose, for a contradiction, that D has an adjoint D^*. Then for all $f, g \in P(\mathbb{R})$, using Integration by Parts, we have

$$\langle D^* g, f\rangle = \langle f, D^* g\rangle = \langle Df, g\rangle = \langle f', g\rangle = \int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$$

In particular, taking g to be the constant polynomial $g(x) = 1$, so $g'(x) = 0$, we find that $\langle D^*1, f\rangle = f(b) - f(a)$ for all $f \in P(\mathbb{R})$. But, as we just showed, there is no such polynomial $D^*1 \in P(\mathbb{R})$.