MATH 245 Linear Algebra 2, Exercises for Chapter 7

1: Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $W=\mathbb{F}^\infty$ with its standard inner product. Let $U=\left\{a=(a_1,a_2,\cdots)\in W \,\middle|\, \sum_{k=1}^{\infty} a_k=0\right\}$. Let $\mathcal{S}=\{e_1,e_2,e_3,\cdots\}$ be the standard basis for W, where $e_n=(e_{n,1},e_{n,2},e_{n,3},\cdots)$ with $e_{n,k}=\delta_{n,k}$.

(a) Recall that the *annihilator* of U in W^* is the vector space $U^0=\left\{f\in W^* \,\middle|\, f(a)=0 \text{ for all } a\in U\right\}$. Show that $\dim(U^0)=1$.

(b) Let $\mathcal{F}=\left\{f_1,f_2,f_3,\cdots\right\}$ where $f_n\in W^*$ is determined by $f_n(e_k)=\delta_{n,k}$. Show that $\mathcal{F}$ is linearly independent but does not span W^*.

(c) Define $E:W\to W^{**}$ by $E(a)(f)=f(a)$, where $a\in W$ and $f\in W^*$. Show that E is 1:1 but not onto.

(d) Define $L:W\to W$ by $L(a)_k=\sum_{i=k}^{\infty} a_i$, where $a\in W$. Show that L has no adjoint.

2: Let $U=P(\mathbb{R})=\mathbb{R}[x]$. Fix $p\in U$. Let $L:U\to U$ be multiplication by p, that is $L(f)=pf$ for all $f\in U$, and let $D:U\to U$ be the differentiation operator, that is $D(f)=f'$ for all $f\in U$.

(a) Show that if we use the inner product on U given by $\left\langle \sum a_i x^i, \sum b_i x^i\right\rangle=\sum a_i b_i$ then both L and D have adjoints.

(b) Show that if we use the inner product given by $\langle f,g\rangle=\int_a^b fg$, then L has an adjoint but D does not. Hint: to show that D does not have an adjoint, you might find it useful to show first that there is no $g\in U$ with the property that $\langle g,f\rangle=f(b)-f(a)$ for every $f\in U$, and then use Integration by Parts.