MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 6

1: Let $u_1=\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$, $u_2=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}$, $u_3=\begin{pmatrix}1\\-3\\2\\1\end{pmatrix}$ and $x=\begin{pmatrix}1\\1\\7\\3\end{pmatrix}$. Let $\mathcal{U}=\{u_1,u_2,u_3\}$ and let $U=\text{Span}\,\mathcal{U}$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A=\big(u_1,u_2,u_3\big)\in M_{4\times 3}$ then use the formula $\text{Proj}_U(x)=At$ where t is the solution to $A^TA\,t=A^Tx$.

Solution: We have

$$A^TA=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}=\begin{pmatrix}3&3&2\\3&6&1\\2&1&15\end{pmatrix}$$

$$A^Tx=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1\\1\\7\\3\end{pmatrix}=\begin{pmatrix}5\\10\\15\end{pmatrix}$$

$$\big(A^TA\big|A^Tx\big)=\left(\begin{array}{ccc|c}3&3&2&5\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&0&40&40\\0&3&-41&-35\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&3&-41&-35\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&3&0&6\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&1&0&2\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&-1\\0&1&0&2\\0&0&1&1\end{array}\right)$$

and so

$$t=\begin{pmatrix}-1\\2\\1\end{pmatrix}\text{ and }\text{Proj}_U(x)=At=\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{U}$ to obtain an orthogonal basis $\mathcal{V}=\{v_1,v_2,v_3\}$ for U, then use the formula $\text{Proj}_U(x)=\sum_{i=1}^{3}\frac{x\cdot v_i}{|v_i|^2}\,v_i$.

Solution: We let

$$v_1=u_1=\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$$

$$v_2=u_2-\frac{u_2\cdot v_1}{|v_1|^2}\,v_1=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}-\frac{3}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$v_3=u_3-\frac{u_3\cdot v_1}{|v_1|^2}\,v_1-\frac{u_3\cdot v_2}{|v_2|^2}\,v_2=\begin{pmatrix}1\\-3\\2\\1\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\-8\\4\\6\end{pmatrix}=\frac{2}{3}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Then

$$\text{Proj}_U(x)=\frac{x\cdot v_1}{|v_1|^2}\,v_1-\frac{x\cdot v_2}{|v_2|^2}\,v_2-\frac{x\cdot v_3}{|v_3|^2}\,v_3=\frac{5}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{5}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}+\frac{20}{30}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}12\\-3\\9\\6\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

(c) Find $w \in \mathbb{R}^4$ so that $\{w\}$ is a basis for $U^\perp$, then calculate $\text{Proj}_U(x) = x - \text{Proj}_w(x) = x - \dfrac{x \bullet w}{|w|^2}\, w$.

Solution: Let $A = (u_1, u_2, u_3) \in M_{4\times 3}$. We wish to find a basis for $U^\perp = (\text{ColA})^\perp = (\text{RowA}^\text{T})^\perp = \text{NullA}^\text{T}$. We have

$$A^T = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 0 \\ 1 & -3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

so we can take $w = (-3, 2, 4, 1)^T$. Thus

$$\text{Proj}_U(x) = x - \frac{x \bullet w}{|w|^2}\, w = \begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix} - \frac{30}{30} \begin{pmatrix} -3 \\ 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

2: (a) Let $W = P_2(\mathbb{R})$ with the inner product given by $\langle f, g\rangle = \sum_{k=0}^{2} f(k)g(k)$. Let $U \subseteq W$ be the subspace $U = \text{Span}\{1 + x\,,\ 5 - 2x + x^2\}$. Find $\text{Proj}_U(x^2)$.

Solution: Let $f_1(x) = 1+x$ and $f_2(x) = 5-2x+x^2$ and let $h(x) = x^2$. We apply the Gram-Schmidt procedure to the basis $\{f_1, f_2\}$, using the inner product $\{f, g\rangle = f(0)g(0) + f(1)g(1) + f(2)g(2)$ to obtain the orthogonal basis $\{g_1, g_2\}$, where

$$\begin{aligned} g_1 &= f_1 = 1 + x \\ g_2 &= f_2 - \frac{\langle f_2, g_1\rangle}{|g_1|^2}\, g_1 = (5 - 2x + x^2) - \frac{5\cdot 1 + 4\cdot 2 + 5\cdot 3}{0^2 + 1^2 + 2^2}\,(1+x) = (5 - 2x + x^2) - 2(1+x) = 3 - 4x + x^2. \end{aligned}$$

The required projection is given by

$$\begin{aligned} \text{Proj}_U(x^2) &= \frac{\langle x^2, g_1\rangle}{|g_1|^2}\, g_1 + \frac{\langle x^2, g_2\rangle}{|g_2|^2}\, g_2 = \frac{0\cdot 1 + 1\cdot 2 + 4\cdot 3}{1^2 + 2^2 + 3^2}(1+x) + \frac{0\cdot 3 + 1\cdot 0 + 4\cdot(-1)}{3^2 + 0^2 + (-1)^2}(3 - 4x + x^2) \\ &= (1+x) - \tfrac{2}{5}(3 - 4x + x^2) = \tfrac{1}{5}(-1 + 13x - 2x^2). \end{aligned}$$

(b) Let $W = \mathcal{C}^0\big([-1,1], \mathbb{R}\big)$ with the inner product given by $\langle f, g\rangle = \int_{-1}^{1} fg$. Using the orthogonal basis $\big\{1, x, x^2 - \frac{1}{3}\big\}$ for $P_2(\mathbb{R}) \subseteq W$, find the polynomial $f \in P_2(\mathbb{R})$ which minimizes $\int_{-1}^{1} \big(f(x) - x^{2/3}\big)^2 dx$.

Solution: To minimize $\int_{-1}^{1} \big(f(x) - x^{2/3}\big)^2 = d\big(f(x), x^{2/3}\big)^2$ we must take

$$\begin{aligned} f = \text{Proj}_{P_2(\mathbb{R})}(x^{2/3}) &= \frac{\langle x^{2/3}, 1\rangle}{|1|^2} \cdot 1 + \frac{\langle x^{2/3}, x\rangle}{|x|^2} \cdot x + \frac{\langle x^{\frac{2}{3}}, x^2 - \frac{1}{3}\rangle}{|x^2 - \frac{1}{3}|^2} \cdot (x - \tfrac{1}{3}) \\ &= \frac{\int_{-1}^{1} x^{2/3}}{\int_{-1}^{1} 1} \cdot 1 + \frac{\int_{-1}^{1} x^{5/3}}{\int_{-1}^{1} x^2} \cdot x + \frac{\int_{-1}^{1} x^{8/3} - \frac{1}{3}x^{2/3}}{\int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}} \cdot (x^2 - \tfrac{1}{3}) \\ &= \tfrac{6/5}{2} \cdot 1 + \tfrac{0}{2/3} \cdot x + \frac{2(\frac{3}{11} - \frac{1}{5})}{2(\frac{1}{5} - \frac{2}{9} + \frac{1}{9})} \cdot (x^2 - \tfrac{1}{3}) \\ &= \tfrac{3}{5} + \tfrac{4/55}{4/45}(x^2 - \tfrac{1}{3}) = \tfrac{3}{5} + \tfrac{9}{11}(x^2 - \tfrac{1}{3}) = \tfrac{9}{11}x^2 + \tfrac{18}{55}. \end{aligned}$$

3: (a) Let $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 1 & -4 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$ and $A_4 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find the orthogonal basis for $M_2(\mathbb{R})$ which is obtained by applying the Gram-Schmidt Procedure to the basis $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$.

Solution: Applying the Gram-Scmidt Procedure gives

$$\begin{aligned}
B_1 &= A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \\
B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{|B_1|^2} B_1 = \begin{pmatrix} 1 & 2 \\ 1 & -4 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}, \\
B_3 &= A_3 - \frac{\langle A_3, B_1 \rangle}{|B_1|^2} B_1 - \frac{\langle A_3, B_2 \rangle}{|B_2|^2} B_2 = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \frac{8}{16} \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix}, \\
B_4 &= A_4 - \frac{\langle A_4, B_1 \rangle}{|B_1|^2} B_1 - \frac{\langle A_4, B_2 \rangle}{|B_2|^2} B_2 - \frac{\langle A_4, B_3 \rangle}{|B_3|^2} B_3 \\
&= \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \frac{2}{16} \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} - \frac{3}{12} \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

(b) Let $U = \left\{ x = (x_0, x_1, x_2, \cdots) \in \mathbb{R}^\infty \middle| \sum_{i=0}^{\infty} x_i = 0 \right\}$. Find the orthogonal basis for U which is obtained by applying the Gram-Schmidt Procedure to the basis $\mathcal{A} = \{u_1, u_2, u_3, \cdots\}$ where $u_k = e_k - e_0$.

Solution: Let $\mathcal{B} = \{v_1, v_2, v_3, \cdots\}$ be the basis obtained by applying the Gram-Schmidt Procedure to $\mathcal{A}$. Write

$$s_k = \sum_{i=0}^{n-1} e_i = (1, 1, \cdots, 1, 0, 0, \cdots).$$

We claim that

$$v_k = e_k - \tfrac{1}{k} s_k = e_k - \tfrac{1}{k} \sum_{i=0}^{k-1} e_i = \left(-\tfrac{1}{k}, -\tfrac{1}{k}, \cdots, -\tfrac{1}{k}, 1, 0, 0, \cdots \right) \text{ for all } k \geq 1.$$

We have $v_1 = u_1 = e_1 - e_0$, so the claim holds when $k = 1$. Let $n \geq 2$ and suppose the claim holds for all $k < n$. For $k < n$ we have

$$\begin{aligned}
\langle u_n, v_k \rangle &= \left\langle e_n - e_0 \,,\, e_k - \tfrac{1}{k} \sum_{i=1}^{k} e_i \right\rangle = \tfrac{1}{k} \text{ , and} \\
|v_k|^2 &= k \cdot \tfrac{1}{k^2} + 1 = \tfrac{k+1}{k}
\end{aligned}$$

and so

$$\begin{aligned}
v_n &= u_n - \sum_{k=1}^{n-1} \frac{\langle u_n, v_k \rangle}{|v_k|^2} v_k = u_n - \sum_{k=1}^{n-1} \tfrac{1}{k+1} v_k = u_n - \sum_{k=1}^{n-1} \tfrac{1}{k+1} \left(e_k - \tfrac{1}{k} s_k \right) \\
&= u_n - \sum_{k=1}^{n-1} \tfrac{1}{k+1} e_k + \sum_{k=1}^{n-1} \left(\tfrac{1}{k} - \tfrac{1}{k+1} \right) s_k = u_n + \sum_{k=1}^{n-1} \tfrac{1}{k} s_k - \sum_{k=1}^{n-1} \tfrac{1}{k+1} (e_k + s_k) \\
&= u_n + \sum_{k=1}^{n-1} \tfrac{1}{k} s_k - \sum_{k=1}^{n-1} \tfrac{1}{k+1} s_{k+1} = u_n + \sum_{k=1}^{n-1} \tfrac{1}{k} s_k - \sum_{l=2}^{n} \tfrac{1}{l} s_l \\
&= u_n + s_1 - \tfrac{1}{n} s_n = (e_n - e_0) + e_0 - \tfrac{1}{n} s_n = e_n - \tfrac{1}{n} s_n,
\end{aligned}$$

as required.

4: (a) Use an orthogonal projection to find $f\in P_2(\mathbb{R})$ which minimizes $\displaystyle\int_0^2 \left(f(x)-\sqrt{2x-x^2}\right)^2 dx$.

Solution: To simplify calculations, we translate 1 unit to the left. Let $g(x)=\sqrt{2x-x^2}$, $h(x)=f(x+1)$ and $k(x)=g(x+1)=\sqrt{1-x^2}$ and note that

$$\int_{x=0}^2 \big(f(x)-g(x)\big)^2 dx = \int_{x=-1}^1 \big(h(x)-k(x)\big)^2 dx.$$

We work in the vector space $\mathcal{C}^0\big([-1,1],\mathbb{R}\big)$ with its standard inner product. To minimize $\int_{-1}^1 (h(x)-k(x))^2 dx$, which is equal to $|h-k|^2$, we must choose $h=\text{Proj}_{P_2}(k)$. To find this projection, we use an orthogonal basis for P_2. In Example 6.21 in the Lecture Notes, the Gram-Schmidt Procedure was applied to the standard basis $\{1,x,x^2\}$ for P_2 to obtain the orthogonal basis $\{q_0,q_1,q_2\}$ with $q_0(x)=1$, $q_1(x)=x$ and $q_2(x)=x^2-\frac{1}{3}$. The calculations done in that example also show that $|q_0|^2=2$, $|q_1|^2=\frac{2}{3}$ and $|q_2|^2=\frac{8}{45}$. Note that

$$\langle k,q_0\rangle = \big\langle \sqrt{1-x^2},1\big\rangle = \int_{-1}^1 \sqrt{1-x^2}\,dx = \tfrac{\pi}{2}$$

since the integral calculates the area under the semicircle $y=\sqrt{1-x^2}$. Also note that

$$\langle k,q_1\rangle = \big\langle \sqrt{1-x^2},x\big\rangle = \int_{-1}^1 x\sqrt{1-x^2}\,dx = 0$$

by symmetry, since $x\sqrt{1-x^2}$ is an odd function. To find $\langle k,q_3\rangle$ we need to find $\int_{-1}^1 x^2\sqrt{1-x^2}\,dx$. We make the substitution $\sin\theta=x$ so that $\cos\theta=\sqrt{1-x^2}$ and $\cos\theta\,d\theta=dx$ to get

$$\begin{aligned}\int_{x=-1}^1 x^2\sqrt{1-x^2}\,dx &= \int_{\theta=-\pi/2}^{\pi/2} \sin^2\theta\cos^2\theta\;d\theta = \int_{\theta=-\pi/2}^{\pi/2} \frac{\sin^2 2\theta}{4}\,d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \frac{1-\cos 4\theta}{8}\,d\theta = \Big[\tfrac{1}{8}\theta-\tfrac{1}{32}\sin 4\theta\Big]_{-\pi/2}^{\pi/2} = \tfrac{\pi}{8}\end{aligned}$$

and so

$$\langle k,q_2\rangle = \big\langle \sqrt{1-x^2},x^2-\tfrac{1}{3}\big\rangle = \int_{-1}^1 x^2\sqrt{1-x^2}\,dx - \tfrac{1}{3}\int_{-1}^1 \sqrt{1-x^2}\,dx = \tfrac{\pi}{8}-\tfrac{\pi}{6} = -\tfrac{\pi}{24}.$$

Thus in order to minimize $|h-k|^2$ we must choose

$$\begin{aligned}h(x)=\text{Proj}_{P_2}(k) &= \frac{\langle k,q_0\rangle}{|q_0|^2}\,q_0+\frac{\langle k,q_1\rangle}{|q_1|^2}\,q_1+\frac{\langle k,q_2\rangle}{|q_2|^2}\,q_2 \\ &= \tfrac{\pi/2}{2}\cdot 1+\tfrac{0}{2/3}\cdot x+\tfrac{-\pi/24}{8/45}\cdot\big(x^2-\tfrac{1}{3}\big) = \tfrac{\pi}{4}-\tfrac{15\pi}{64}\big(x^2-\tfrac{1}{3}\big) = \tfrac{\pi}{64}\big(21-15x^2\big)\end{aligned}$$

and hence we must choose

$$f(x)=h(x-1)=\tfrac{\pi}{64}\big(6+30x-15x^2\big).$$

(b) Let $a,b\in\mathbb{R}$ with $a<b$ and let $W=\mathcal{C}^0\big([a,b],\mathbb{R}\big)$ with the inner product given by $\langle f,g\rangle=\int_a^b fg$. Suppose $\{p_0,p_1,\cdots,p_n\}$ is an orthonormal basis for $P_n(\mathbb{R})\subseteq W$. For each k, write $p_k(x)=\sum\limits_{i=0}^n a_{k,i}x^i$ and let $A\in M_{n+1}(\mathbb{R})$ with $A_{ki}=a_{ki}$. Let $b=(b_0,b_1,\cdots,b_n)^T\in\mathbb{R}^{n+1}$. Given that $f\in W$ with $\int_a^b x^i f(x)\,dx=b_i$ for $0\le i\le n$, find a formula, in terms of A and b, for the minimum possible value for $\int_a^b f(x)^2dx$.

Solution: Since $\langle f,x^i\rangle=\int_a^b x^i f(x)\,dx=b_i$ for all i, we have $\langle f,p_k\rangle=\big\langle f,\sum\limits_{i=1}^n a_{ki}x^i\big\rangle=\sum\limits_{i=1}^n a_{ki}\langle f,x^i\rangle=\sum\limits_{i=1}^n a_{ki}b_i$ which is equal to the k^{th} entry of Ab. Thus we have $Ab=\big(\langle f,p_0\rangle,\langle f,p_1\rangle,\cdots,\langle f,p_n\rangle\big)^T$. When $f=u+v$ with $u\in P_n(\mathbb{R})$ and $v\in P_n(\mathbb{R})^\perp$, we have $|f|^2=|u+v|^2=|u|^2+2\,u\cdot v+|v|^2=|u|^2+|v|^2$ because $u\cdot v=0$, and so $|f|\le|u|$ with $|f|=|u|\iff v=0\iff f=u$. Thus in order to minimize $|f|$ we must have $f=u=\text{Proj}_{P_2(\mathbb{R})}(f)$. Thus the minimum possible value for $\int_a^b f(x)^2dx=|f|^2$ is equal to

$$|u|^2=\sum_{i=0}^n \big|\langle f,p_k\rangle\big|^2=\big|Ab\big|^2.$$