MATH 245 Linear Algebra 2, Exercises for Chapter 6

1: Let $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix}$. Let $\mathcal{A} = \{u_1, u_2, u_3\}$ and let $U = \text{Span}\,\mathcal{A}$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A = \big(u_1, u_2, u_3\big) \in M_{4\times 3}$ then use the formula $\text{Proj}_U(x) = At$ where t is the solution to $A^TA\,t = A^Tx$.

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{A}$ to obtain an orthogonal basis $\mathcal{B} = \{v_1, v_2, v_3\}$ for U, then use the formula $\text{Proj}_U(x) = \sum_{i=1}^{3} \frac{x \bullet v_i}{|v_i|^2}\, v_i$.

(c) Find $w \in \mathbb{R}^4$ so that $\{w\}$ is a basis for $U^\perp$, then calculate $\text{Proj}_U(x) = x - \text{Proj}_w(x) = x - \frac{x \bullet w}{|w|^2}\, w$.

2: (a) Let $W = P_2(\mathbb{R})$ with the inner product given by $\langle f, g\rangle = \sum_{k=0}^{2} f(k)g(k)$. Let $U \subseteq W$ be the subspace $U = \text{Span}\big\{1 + x\,,\ 5 - 2x + x^2\big\}$. Find $\text{Proj}_U(x^2)$.

(b) Let $W = \mathcal{C}^0\big([-1,1],\mathbb{R}\big)$ with the inner product given by $\langle f, g\rangle = \int_{-1}^{1} fg$. Using the orthogonal basis $\big\{1, x, x^2 - \frac{1}{3}\big\}$ for $P_2(\mathbb{R}) \subseteq W$, find the polynomial $f \in P_2(\mathbb{R})$ which minimizes $\int_{-1}^{1} \big(f(x) - x^{2/3}\big)^2 dx$.

3: (a) Let $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 1 & -4 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$ and $A_4 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find the orthogonal basis for $M_2(\mathbb{R})$ which is obtained by applying the Gram-Schmidt Procedure to the basis $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$.

(b) Let $U = \Big\{x = (x_0, x_1, x_2, \cdots) \in \mathbb{R}^\infty \Big| \sum_{i=0}^{\infty} x_i = 0\Big\}$. Find the orthogonal basis for U which is obtained by applying the Gram-Schmidt Procedure to the basis $\mathcal{A} = \{u_1, u_2, u_3, \cdots\}$ where $u_k = e_k - e_0$.

4: (a) Use an orthogonal projection to find $f \in P_2(\mathbb{R})$ which minimizes $\int_0^2 \Big(f(x) - \sqrt{2x - x^2}\Big)^2 dx$.

(b) Let $a, b \in \mathbb{R}$ with $a < b$ and let $W = \mathcal{C}^0\big([a,b],\mathbb{R}\big)$ with the inner product given by $\langle f, g\rangle = \int_a^b fg$. Suppose $\big\{p_0, p_1, \cdots, p_n\big\}$ is an orthonormal basis for $P_n(\mathbb{R}) \subseteq W$. For each k, write $p_k(x) = \sum_{i=0}^{n} a_{k,i}x^i$ and let $A \in M_{n+1}(\mathbb{R})$ with $A_{ki} = a_{ki}$. Let $b = (b_0, b_1, \cdots, b_n)^T \in \mathbb{R}^{n+1}$. Given that $f \in W$ with $\int_a^b x^i f(x)\,dx = b_i$ for $0 \le i \le n$, find a formula, in terms of A and b, for the minimum possible value for $\int_a^b f(x)^2 dx$.