MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 5

1: (a) For $u=(u_1,u_2,\cdots,u_n)^T\in\mathbb{C}^n$, define $|u|_1=\sum_{i=1}^n|u_i|$. Show that $|\ |_1$ is a norm on $\mathbb{C}^n$ but that there is no inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$ such that $|u|_1=\sqrt{\langle u,u\rangle}$ for all $u\in\mathbb{C}^n$.

Solution: We note that $|\ |_1$ is a norm on $\mathbb{R}^n$ because for all $x,y\in\mathbb{R}^n$ and all $t\in\mathbb{R}$ we have

1. $|tx|_1=\sum_{i=1}^n|tx_i|=|t|\sum_{i=1}^n|x_i|=|t|\,|x|_1$,
2. $|x|_1=\sum_{i=1}^n|x_i|\geq 0$ with $|x|_1=0\iff\sum_{i=1}^n|x_i|=0\iff|x_i|=0$ for all $i\iff x=0$, and
3. $|x+y|_1=\sum_{i=1}^n|x_i+y_i|\leq\sum_{i=1}^n\big(|x_i|+|y_i|\big)=\sum_{i=1}^n|x_i|+\sum_{i=1}^n|y_i|=|x|_1+|y|_1$.

Suppose, for a contradiction, that there was an inner product $\langle\ ,\ \rangle$ such that $|x|_1=\sqrt{\langle x,x\rangle}$ for all $x\in\mathbb{R}^n$. Then by the polarization identity we would have

$$\left\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle=\tfrac12\left(\left|\begin{pmatrix}1\\1\end{pmatrix}\right|_1^2-\left|\begin{pmatrix}1\\0\end{pmatrix}\right|_1^2-\left|\begin{pmatrix}0\\1\end{pmatrix}\right|_1^2\right)=\tfrac12(4-1-1)=1\ ,\text{ and}$$
$$\left\langle\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle=\tfrac12\left(\left|\begin{pmatrix}-1\\1\end{pmatrix}\right|_1^2-\left|\begin{pmatrix}-1\\0\end{pmatrix}\right|_1^2-\left|\begin{pmatrix}0\\1\end{pmatrix}\right|_1^2\right)=\tfrac12(4-1-1)=1\,,$$

but this is not possible since by linearity, we would also have

$$\left\langle\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle=\left\langle-\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle=-\left\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle.$$

(b) For $A\in M_{l\times m}(\mathbb{C})$, define $|A|=\max_{u\in\mathbb{C}^m,|u|=1}|Au|$. Show that $|\ |$ is a norm on $M_{l\times m}(\mathbb{C})$ and show that for $A\in M_{l\times m}(\mathbb{C})$ and $B\in M_{m\times n}(\mathbb{C})$, we have $|AB|\leq|A|\,|B|$. (You may assume, without proof, that the maximum $\max_{u\in\mathbb{C}^m,|u|=1}|Au|$ exists and is finite. This follows from the Extreme Value Theorem, since the set $S=\big\{u\in\mathbb{C}^m\big||u|=1\big\}$ is compact and the map $g(u)=|Au|$ is continuous on S).

Solution: For $A,B\in M_{l\times m}(\mathbb{C})$ and for $t\in\mathbb{C}$ we have

$$|tA|=\max_{|x|=1}|t\,Ax|=\max_{|x|=1}|t|\,|Ax|=|t|\max_{|x|=1}|Ax|=|t|\,|A|\text{ and}$$
$$|A+B|=\max_{|x|=1}\big|Ax+Bx\big|\leq\max_{|x|=1}\big(|Ax|+|Bx|\big)\leq\max_{|x|=1}|Ax|+\max_{|x|=1}|Bx|=|A|+|B|\,,$$

and we have $|A|\geq 0$ with

$$|A|=0\iff\max_{|x|=1}|Ax|=0\iff Ax=0\text{ for all }x\in\mathbb{C}^m\text{ with }|x|=1\iff A=0,$$

indeed if $A=0$ the of course $Ax=0$ for all $x\in\mathbb{C}^m$ with $|x|=1$, and conversely if $Ax=0$ for all $x\in\mathbb{C}^m$ with $|x|=1$ then, in particular we have $Ae_i=0$ for all indices i and so $A=(Ae_1,Ae_2,\cdots,Ae_l)=0$. Thus $|\ |$ is a norm on $M_{l\times n}(\mathbb{C})$. Now let $A\in M_{l\times m}(\mathbb{C})$ and $B\in M_{m\times n}(\mathbb{C})$. Then

$$|AB|=\max_{x\in\mathbb{C}^n,|x|=1}|ABx|=\max_{x\in\mathbb{C}^n,|x|=1}\frac{|ABx|}{|Bx|}\,|Bx|=\max_{x\in\mathbb{C}^n,|x|=1}\left|A\big(\tfrac{Bx}{|Bx|}\big)\right||Bx|$$
$$\leq\left(\max_{x\in\mathbb{C}^n,|x|=1}\left|A\big(\tfrac{Bx}{|Bx|}\big)\right|\right)\left(\max_{x\in\mathbb{C}^n,|x|=1}|Bx|\right)\leq\left(\max_{y\in\mathbb{C}^m,|y|=1}|Ay|\right)\left(\max_{x\in\mathbb{C}^n,|x|=1}|Bx|\right)=|A|\,|B|.$$

2: (a) A matrix $A \in M_n(\mathbb{C})$ is called **Hermitian** (or **self-adjoint**) when $A^* = A$, and **positive-definite** when $A^* = A$ with $u^*Au > 0$ for all $0 \neq u \in \mathbb{C}^n$. Show that the following matrix A is positive-definite:

$$A = \begin{pmatrix} 2 & 1 & i \\ 1 & 3 & 2 \\ -i & 2 & 3 \end{pmatrix}.$$

Solution: It is clear that $A^* = A$. For $u \in \mathbb{C}^3$ we have

$$\begin{aligned} u^*Au &= \left(\overline{u_1}, \overline{u_2}, \overline{u_3}\right) \begin{pmatrix} 2 & 1 & i \\ 1 & 3 & 2 \\ -i & 2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \left(\overline{u_1}, \overline{u_2}, \overline{u_3}\right) \begin{pmatrix} 2u_1 + u_2 + i\,u_3 \\ u_1 + 3u_2 + 2u_3 \\ -i + 2u_2 + 3u_3 \end{pmatrix} \\ &= 2u_1\overline{u_1} + u_2\overline{u_1} + i\,u_3\overline{u_1} + u_1\overline{u_2} + 3u_2\overline{u_2} + 2u_3\overline{u_2} - i\,u_1\overline{u_3} + 2u_2\overline{u_3} + 3u_3\overline{u_3} \\ &= \left(u_1\overline{u_1} + u_2\overline{u_1} + u_1\overline{u_2} + u_2\overline{u_2}\right) + \left(u_1\overline{u_1} + i\,u_3\overline{u_1} - i\,u_1\overline{u_3} + u_3\overline{u_3}\right) + 2\left(u_2\overline{u_2} + u_3\overline{u_2} + u_2\overline{u_3} + u_3\overline{u_3}\right) \\ &= |u_1 + u_2|^2 + |u_1 + i\,u_3|^2 + 2|u_2 + u_3|^2 \geq 0 \end{aligned}$$

with

$$u^*Au = 0 \iff u_1 + u_2 = u_1 + i\,u_3 = u_2 + u_3 = 0 \iff u_1 = u_2 = u_3 = 0.$$

(b) Let U be an n-dimensional vector space over $\mathbb{C}$ and let $\mathcal{A} = (u_1, u_2, \cdots, u_n)$ be an ordered basis for U. Show that given an inner product $\langle\ ,\ \rangle$ on U, there exists a unique matrix $A \in M_n(\mathbb{C})$ (which we call the **matrix of the inner product** with respect to the basis $\mathcal{A}$) such that $\langle x, y\rangle = [y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}}$ for all $x, y \in U$ and this matrix A is Hermitian and positive-definite, and show, conversely, that given a Hermitian positive-definite matrix $A \in M_n(\mathbb{C})$, we can define an inner-product on U by $\langle x, y\rangle = [y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}}$ for $x, y \in U$.

Solution: Let $\langle\ ,\ \rangle$ be an inner product on U. Suppose that there exists a matrix $A \in M_n(\mathbb{C})$ with the property that $\langle x, y\rangle = [y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}}$ for all $x, y \in U$. Then in particular, for all indices k and l we must have

$$\langle u_l, u_k\rangle = [u_k]_{\mathcal{A}}^* A [u_l]_{\mathcal{A}} = e_k A a_l = A_{k,l}.$$

On the other hand, when we define $A \in M_r(\mathbb{C})$ to be the matrix with entries $A_{k,l} = \langle u_l, u_k\rangle$, for $x = \sum_{i=1}^n s_i u_i$ and $y = \sum_{j=1}^n t_j u_j$ we have

$$\langle x, y\rangle = \Big\langle \sum_{i=1}^n s_i u_i, \sum_{j=1}^n t_j u_j \Big\rangle = \sum_{i=1}^n \sum_{j=1}^n s_i \overline{t_j} \langle u_i, u_j\rangle = \sum_{i=1}^n \sum_{j=1}^n \overline{t_j} A_{j,i} s_i = t^* A s = [y]_{\mathcal{A}}^* A [x]_{\mathcal{A}}.$$

Note that A is Hermitian because for all indices k, l we have $(A^*)_{k,l} = \overline{A_{l,k}} = \overline{\langle u_k, u_l\rangle} = \langle u_l, u_k\rangle = A_{k,l}$ and A is positive-definite because given $0 \neq t \in \mathbb{C}^n$, if we let $x = \sum_{i=1}^n t_i u_i$ then we have $[x]_{\mathcal{A}} = t$ and $x \neq 0$, and so $t^*At = [x]_{\mathcal{A}}^* A [x]_{\mathcal{A}} = \langle x, x\rangle > 0$.

Conversely, let $A \in M_n(\mathbb{C})$ be Hermitian and positive-definite and define $\langle\ ,\ \rangle$ by $\langle x, y\rangle = [y]_{\mathcal{A}}^* A [x]_{\mathcal{A}}$ for $x, y \in \mathbb{C}^n$. Then $\langle\ ,\ \rangle$ is an inner product because

1. $\langle x, y + z\rangle = [y + z]_{\mathcal{A}}^* A [x]_{\mathcal{A}} = [y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}} + [z]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}} = \langle x, y\rangle + \langle x, z\rangle$, and
   $\langle x, ty\rangle = [ty]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}} = \left(t[y]_{\mathcal{A}}\right)^* A\,[x]_{\mathcal{A}} = \overline{t}[y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}} = \overline{t}\langle x, y\rangle$, and similarly
   $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle tx, y\rangle = t\langle x, y\rangle$,
2. $\overline{\langle x, y\rangle} = (\langle x, y\rangle)^* = \left([y]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}}\right)^* = [x]_{\mathcal{A}}^* A^*\,[y]_{\mathcal{A}} = [x]_{\mathcal{A}}^* A\,[y]_{\mathcal{A}} = \langle y, x\rangle$, and
3. for $0 \neq x \in \mathbb{C}^n$ we have $[x]_{\mathcal{A}} \neq 0$ so $\langle x, x\rangle = [x]_{\mathcal{A}}^* A\,[x]_{\mathcal{A}} > 0$.

(c) Let U be an n-dimensional inner product space over $\mathbb{C}$, let $\mathcal{A} = (u_1, \cdots, u_n)$ and $\mathcal{B} = (v_1, \cdots, v_n)$ be two ordered bases for U, let A and B be the matrices of the inner product with respect to the bases $\mathcal{A}$ and $\mathcal{B}$, and let $P = [I]_{\mathcal{B}}^{\mathcal{A}} = \left([u_1]_{\mathcal{B}}, \cdots, [u_n]_{\mathcal{B}}\right)$ be the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$. Find a formula for B in terms of A and P.

Solution: For all indices k, l we have

$$\begin{aligned} B_{k,l} = \langle v_l, v_k\rangle = [v_k]_{\mathcal{A}}^* A [v_l]_{\mathcal{A}} &= \left([I]_{\mathcal{A}}^{\mathcal{B}}[v_k]_{\mathcal{B}}\right)^* A \left([I]_{\mathcal{A}}^{\mathcal{B}}[v_l]_{\mathcal{B}}\right) \\ &= \left(P^{-1}e_k\right)^* A \left(P^{-1}e_l\right) = {e_k}^*(P^{-1})^* A P^{-1} e_l = \left((P^{-1})^* A P^{-1}\right)_{k,l} \end{aligned}$$

and so $B = (P^{-1})^* A P^{-1}$.

3: For $z = x + iy$ with $x, y \in \mathbb{R}$ we define $e^z = e^x \cos y + i\, e^x \sin y$ and then we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}\ ,\ \cos z = \frac{e^{iz} + e^{-iz}}{2}\ ,\ \sinh z = \frac{e^z - e^{-z}}{2}\ ,\ \cosh z = \frac{e^z + e^{-z}}{2}.$$

(a) Show that for $x, y \in \mathbb{R}$ we have $\cos(x + iy) = \cos x \cosh y - i\, \sin x \sinh y$.

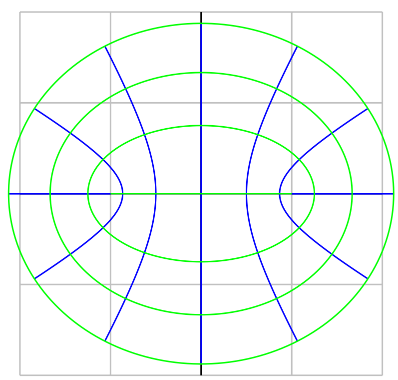
Solution: We have

$$\begin{aligned}\cos(x+iy) &= \tfrac{1}{2}\big(e^{i(x+iy)} + e^{-i(x+iy)}\big) = \tfrac{1}{2}\big(e^{-y+ix} + e^{y-ix}\big) = \tfrac{1}{2}\big(e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)\big) \\ &= \cos x \cdot \tfrac{1}{2}(e^y + e^{-y}) - i\, \sin x \cdot \tfrac{1}{2}(e^y - e^{-y}) = \cos x \cosh y - i\, \sin x \sinh y.\end{aligned}$$

(b) Draw a fairly accurate sketch of the images of the lines $x = \alpha$ for $\alpha \in \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$ and the images of the lines $y = \beta$ for $\beta \in \left\{0, \pm\ln 2, \pm\ln 3, \pm\ln 4\right\}$ under the map $w = f(z) = \cos z$.

Solution: The line $x = \alpha$ is given parametrically by $x + iy = \alpha + it$, and it is mapped to the curve given by $u + iv = \cos(\alpha + it) = \cos\alpha \cosh t + i\, \sin\alpha \sinh t$. When $\alpha = 0$ we get the curve $u + iv = \cosh t + i\,0$ which follows the u-axis with $u \geq 1$ (moving to the left then to the right). When $\alpha = \frac{\pi}{2}$ we get the curve $u + iv = 0 - i\, \sinh t$ which follows the v-axis (downwards). When $\alpha = \pi$ we get the curve $u + iv = -\cosh t + i\,0$ which follows the u-axis with $u \leq -1$ (moving to the right then to the left). For the other values of α note that $\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin\alpha} = \cosh^2\alpha - \sinh^2\alpha = 1$ and so the curve follows the hyperbola $\frac{u^2}{\cos^2\alpha} + \frac{v^2}{\sin^2\alpha} = 1$ (the asymptotes of this hyperbola make an angle α with the u-axis).

The line $y = \beta$ is given parametrically by $x + iy = t + i\beta$, and it is mapped to the curve given by $u + iv = \cos(t + i\beta) = \cos t \cosh\beta - i\, \sin t \sinh\beta$. When $\beta = 0$ we get the curve $u + iv = \cos t + i\,0$ which follows the u-axis with $-1 \leq u \leq 1$ (moving back and forth). For all other values of β note that we have $\frac{u^2}{\cosh^2\beta} + \frac{v^2}{\sinh^2\beta} = \cos^2 t + \sin^2 t = 1$ and so the curve follows the ellipse $\frac{u^2}{\cosh^2\beta} + \frac{v^2}{\sinh^2\beta} = 1$ (which has semi-major and semi-minor axes $\cosh\beta$ and $|\sinh\beta|$).



(c) Recall that the **complex-valued angle** between u and v in $\mathbb{C}^n$ is the unique complex number $\theta = \alpha + i\beta$ with $0 < \alpha < \pi$ such that $\cos\theta = \frac{\langle u, v\rangle}{|u||v|}$. Find the complex-valued angle between the vectors $u = (1, 0, i)^T$ and $v = (3 + i, 2i, 3 + 3i)^T$ in $\mathbb{C}^3$.

Solution: We have

$$\frac{\langle u, v\rangle}{|u||v|} = \frac{v^* u}{|u||v|} = \frac{(3 - i) + i(3 - 3i)}{\sqrt{2}\sqrt{32}} = \frac{6 + 2i}{8} = \frac{3 + i}{4}.$$

For $w = \frac{3+i}{4}$ and $\theta = \alpha + i\beta$ with $0 \leq \alpha \leq \pi$ we have

$$\begin{aligned}\cos\theta = w &\iff \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{3 + i}{4} \iff 2e^{i\theta} + 2e^{-i\theta} = 3 + i \iff 2(e^{i\theta})^2 - (3 + i)(e^{i\theta}) + 2 = 0 \\ &\iff e^{i\theta} = \frac{(3 + i) \pm \sqrt{(3 + i)^2 - 16}}{4} = \frac{(3 + i) \pm \sqrt{-8 + 6i}}{4} = \frac{(3 + i) \pm (1 + 3i)}{4} \\ &\iff e^{i(\alpha + i\beta)} \in \left\{1 + i, \tfrac{1-i}{2}\right\} \iff e^{-\beta} e^{i\alpha} \in \left\{\sqrt{2} e^{i\,\pi/4}, \tfrac{1}{\sqrt{2}}\, e^{-i\,\pi/4}\right\} \\ &\iff e^{-\beta} = \sqrt{2} \text{ and } \alpha = \tfrac{\pi}{4}\ \big(\text{since } 0 \leq \alpha \leq \pi\big) \iff \theta = \tfrac{\pi}{4} - i\, \ln\sqrt{2}.\end{aligned}$$

Thus the complex angle between u and v is $\theta = \cos^{-1}\frac{\langle u, v\rangle}{|u||v|} = \cos^{-1}\frac{3+i}{4} = \frac{\pi}{4} - i\, \ln\sqrt{2}$.