MATH 245 Linear Algebra 2, Exercises for Chapter 5

1: (a) For $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{C}^n$, define $|u|_1 = \sum_{i=1}^{n} |u_i|$. Show that $|\ |_1$ is a norm on $\mathbb{C}^n$ but that there is no inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$ such that $|u|_1 = \sqrt{\langle u, u\rangle}$ for all $u \in \mathbb{C}^n$.

(b) For $A \in M_{l\times m}(\mathbb{C})$, define $|A| = \max_{u\in\mathbb{C}^m, |u|=1} |Au|$. Show that $|\ |$ is a norm on $M_{l\times m}(\mathbb{C})$ and show that for $A \in M_{l\times m}(\mathbb{C})$ and $B \in M_{m\times n}(\mathbb{C})$, we have $|AB| \leq |A|\,|B|$. (You may assume, without proof, that the maximum $\max_{u\in\mathbb{C}^m, |u|=1} |Au|$ exists and is finite. This follows from the Extreme Value Theorem, since the set $S = \left\{u \in \mathbb{C}^m \big| |u| = 1\right\}$ is compact and the map $g(u) = |Au|$ is continuous on S).

2: (a) A matrix $A \in M_n(\mathbb{C})$ is called **Hermitian** (or **self-adjoint**) when $A^* = A$, and **positive-definite** when $A^* = A$ with $u^*Au > 0$ for all $0 \neq u \in \mathbb{C}^n$. Show that the following matrix A is positive-definite:

$$A = \begin{pmatrix} 2 & 1 & i \\ 1 & 3 & 2 \\ -i & 2 & 3 \end{pmatrix}.$$

(b) Let U be an n-dimensional vector space over $\mathbb{C}$ and let $\mathcal{A} = (u_1, u_2, \cdots, u_n)$ be an ordered basis for U. Show that given an inner product $\langle\ ,\ \rangle$ on U, there exists a unique matrix $A \in M_n(\mathbb{C})$ (which we call the **matrix of the inner product** with respect to the basis $\mathcal{A}$) such that $\langle x, y\rangle = [y]_{\mathcal{A}}^* A\, [x]_{\mathcal{A}}$ for all $x, y \in U$ and this matrix A is Hermitian and positive-definite, and show, conversely, that given a Hermitian positive-definite matrix $A \in M_n(\mathbb{C})$, we can define an inner-product on U by $\langle x, y\rangle = [y]_{\mathcal{A}}^* A\, [x]_{\mathcal{A}}$ for $x, y \in U$.

(c) Let U be an n-dimensional inner product space over $\mathbb{C}$, let $\mathcal{A} = (u_1, \cdots, u_n)$ and $\mathcal{B} = (v_1, \cdots, v_n)$ be two ordered bases for U, let A and B be the matrices of the inner product with respect to the bases $\mathcal{A}$ and $\mathcal{B}$, and let $P = [I]_{\mathcal{B}}^{\mathcal{A}} = \big([u_1]_{\mathcal{B}}, \cdots, [u_n]_{\mathcal{B}}\big)$ be the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$. Find a formula for B in terms of A and P.

3: For $z = x + iy$ with $x, y \in \mathbb{R}$ we define $e^z = e^x \cos y + i\, e^x \sin y$ and then we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}\ ,\ \cos z = \frac{e^{iz} + e^{-iz}}{2}\ ,\ \sinh z = \frac{e^z - e^{-z}}{2}\ ,\ \cosh z = \frac{e^z + e^{-z}}{2}.$$

(a) Show that for $x, y \in \mathbb{R}$ we have $\cos(x + iy) = \cos x \cosh y - i\ \sin x \sinh y$.

(b) Draw a fairly accurate sketch of the images of the lines $x = \alpha$ for $\alpha \in \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$ and the images of the lines $y = \beta$ for $\beta \in \left\{0, \pm\ln 2, \pm\ln 3, \pm\ln 4\right\}$ under the map $w = f(z) = \cos z$.

(c) Recall that the **complex-valued angle** between u and v in $\mathbb{C}^n$ is the unique complex number $\theta = \alpha + i\beta$ with $0 < \alpha < \pi$ such that $\cos\theta = \frac{\langle u,v\rangle}{|u||v|}$. Find the complex-valued angle between the vectors $u = (1, 0, i)^T$ and $v = (3 + i, 2i, 3 + 3i)^T$ in $\mathbb{C}^3$.