MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 4

1: (a) Let $u=(1,0,1,2)^T$, $v=(2,1,1,3)^T$ and $w=(1,2,0,1)^T$. Find $X(u,v,w)$.

Solution: We have

$$X\begin{pmatrix}1&2&1\\0&1&2\\1&1&0\\2&3&1\end{pmatrix}=\left(-\begin{vmatrix}0&1&2\\1&1&0\\2&3&1\end{vmatrix},\begin{vmatrix}1&2&1\\1&1&0\\2&3&1\end{vmatrix},-\begin{vmatrix}1&2&1\\0&1&2\\2&3&1\end{vmatrix},\begin{vmatrix}1&2&1\\0&1&2\\1&1&0\end{vmatrix}\right)^T=(-1,0,-1,1)^T.$$

(b) Let $u_1,u_2,\cdots,u_{n-2}\in\mathbb{R}^n$ with $\{u_1,u_2,\cdots,u_{n-2}\}$ linearly independent, let $A=(u_1,u_2,\cdots,u_{n-2})$ and let $U=\text{Col(A)}$. Show that for $x\in\mathbb{R}^n$ we have $\text{Proj}_{U^\perp}(x)=\dfrac{-1}{\det(A^TA)}X\big(u_1,\cdots,u_{n-2},X(u_1,\cdots,u_{n-2},x)\big)$.

Solution: Let $y=X(u_1,\cdots,u_{n-2},x)$, $w=X(u_1,\cdots,u_{n-2},y)$ and $v=\frac{-1}{\det(A^TA)}w$. Since we have $w\bullet u_k=X(u_1,\cdots,u_{n-2},y)\bullet u_k=0$ for all indices k, we see that $w\in U^\perp$. Since v is a scalar multiple of w, we also have $v\in U^\perp$. From the formula which expresses $X(u_1,\cdots,u_{n-2},X(v_1,\cdots,v_{n-1}))$ as a linear combination of the vectors u_k (namely Part (7) of Theorem 4.9 in the posted Lecture Notes), for $B=(A,y)$ we have

$$\begin{aligned}w&=X\big(u_1,\cdots,u_{n-2},X(u_1,\cdots,u_{n-2},x\big)\\&=\sum_{i=1}^{n-2}(-1)^{n+i}\det\big((B^TA)^{(i)}\big)u_i\ +(-1)^{n+(n-1)}\det\big((B^TA)^{(n-1)}\big)x\\&=\sum_{i=1}^{n-2}(-1)^{n+i}\det\big((B^TA)^{(i)}\big)u_i\ -\det(A^TA)\,x\text{ and so}\\v&=\frac{-1}{\det(A^TA)}\,w=\sum_{i=1}^{n-2}c_iu_i\ +x\ \text{ where } c_i=-\sum_{i1}^{n-2}(-1)^{n+i}\frac{\det\big((B^TA)^{(i)}\big)}{\det(A^TA)}.\end{aligned}$$

Since $v\in U^\perp$ and $v-x=\sum_{i=1}^{n-2}c_iu_i\in U$ it follows that $v=\text{Proj}_{U^\perp}(x)$.

2: (a) Find $X(u_1,u_2,\cdots,u_{n-1})$, where $u_k = e_k - k\,e_n \in \mathbb{R}^n$, for $k=1,2,\cdots,n-1$.

Solution: Let

$$A=(u_1,u_2,\cdots,u_{n-1})=\begin{pmatrix}1&0&0&&0\\0&1&0&\cdots&0\\0&0&1&&0\\&&\vdots&&\\0&0&0&&1\\-1&-2&-3&\cdots&-(n-1)\end{pmatrix}\in M_{n\times(n-1)}\,.$$

Recall that A^k denotes the matrix obtained from A by removing the k^{th} row. Note that $A^n=I$ so $\det(A^n)=1$, and for $1\le k<n$ we have

$$A^k=\begin{pmatrix}1&&&0&&&\\&\ddots&&\vdots&&&\\&&1&0&&&\\&&&0&1&&\\&&&\vdots&&\ddots&\\&&&0&&&1\\-1&\dots&-(k-1)&-k&-(k+1)&\dots&-(n-1)\end{pmatrix}\in M_{(n-1)\times(n-1)}$$

so expanding along the k^{th} column gives $\det(A^k)=(-1)^{k+n-1}(-k)\det(I)=(-1)^{k+n}k$. Thus

$$X(u_1,u_2,\cdots,u_{n-1})=\sum_{k=1}^{n}(-1)^{k+n}\det(A^k)\,e_k=\left(\sum_{k=0}^{n-1}k\,e_k\right)+e_n=\begin{pmatrix}1\\2\\\vdots\\n-1\\n\end{pmatrix}.$$

(b) Let U and V be hyperspaces in $\mathbb{R}^n$ with $U\neq V$. Let $W=U\cap V$ and note that W is $(n-2)$-dimensional and the spaces $U\cap W^\perp$ and $V\cap W^\perp$ are both 1-dimensional. Let $\{w_1,\cdots,w_{n-2}\}$ be a basis for W, let $\{u\}$ be a basis for $U\cap W^\perp$, and let $\{v\}$ be a basis for $V\cap W^\perp$, and note that $\{w_1,\cdots,w_{n-2},u\}$ is a basis for U and $\{w_1,\cdots,w_{n-2},v\}$ is a basis for V. Let $x=X(w_1,\cdots,w_{n-2},u)$ and $y=X(w_1,\cdots,w_{n-2},v)$, and note that $\{x\}$ and $\{y\}$ are bases for $U^\perp$ and $V^\perp$. Let $A=(w_1,\cdots,w_{n-2})\in M_{n\times(n-2)}(\mathbb{R})$. Use Theorem 4.9 to show that $x\bullet y=(u\bullet v)\det(A^TA)$, $|x|^2=|u|^2\det(A^TA)$ and $|y|^2=|y|^2\det(A^TA)$ and hence provide an alternate proof that $\theta(U^\perp,V^\perp)=\theta(U,V)$.

Solution: By Part 6 of Theorem 4.9, we have

$$\begin{aligned}x\bullet y&=X(w_1,\cdots,w_{n-2},u)\bullet X(w_1,\cdots,w_{n-2},v)\\&=\det\big((A,u)^T(A,v)\big)\\&=\det\begin{pmatrix}A^TA&A^Tv\\u^TA&u^Tv\end{pmatrix}\\&=\det\begin{pmatrix}A^TA&0\\0&u\bullet v\end{pmatrix}\ ,\text{ since } u,v\in W^\perp=\text{Null}A^T\\&=(u\bullet v)\det(A^TA)\end{aligned}$$

and similarly

$$\begin{aligned}|x|^2&=X(w_1,\cdots,w_{n-2},u)\bullet X(w_1,\cdots,w_{n-2},u)=(u\bullet u)\det(A^TA)\ ,\text{ and}\\|y|^2&=X(w_1,\cdots,w_{n-2},v)\bullet X(w_1,\cdots,w_{n-2},v)=(v\bullet v)\det(A^TA)\,.\end{aligned}$$

Thus

$$\frac{x\bullet y}{|x||y|}=\frac{(u\bullet v)\det(A^TA)}{|u|\sqrt{\det(A^TA)}|v|\sqrt{\det(A^TA)}}=\frac{u\bullet v}{|u||v|}$$

and hence

$$\theta(U,V)=\cos^{-1}\frac{|u\bullet v|}{|u||v|}=\cos^{-1}\frac{|x\bullet y|}{|x||y|}=\theta(U^\perp,V^\perp)\,.$$