MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 3

1: Find the least-squares best fit quadratic $f \in P_2(\mathbb{R})$ for the following data points.

x_i	-1	0	1	2	3
y_i	0	2	3	2	-2

Solution: Let $A = \begin{pmatrix} 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_3 & {x_3}^2 \\ 1 & x_4 & {x_4}^2 \\ 1 & x_5 & {x_5}^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 2 \\ -2 \end{pmatrix}$. Then we have

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 15 \\ 5 & 15 & 35 \\ 15 & 35 & 99 \end{pmatrix},$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}, \text{ and}$$

$$\begin{aligned} (A^T A | A^T y) &= \left(\begin{array}{ccc|c} 5 & 5 & 15 & 5 \\ 5 & 15 & 35 & 1 \\ 15 & 35 & 99 & -7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 10 & 20 & -4 \\ 0 & 20 & 54 & -22 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & -\frac{2}{5} \\ 0 & 0 & 14 & -14 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{7}{5} \\ 0 & 1 & 2 & -\frac{2}{5} \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{12}{5} \\ 0 & 1 & 0 & \frac{8}{5} \\ 0 & 0 & 1 & -1 \end{array}\right). \end{aligned}$$

Thus the best fit quadratic is $f(x) = \frac{12}{5} + \frac{8}{5}\,x - x^2$.

2: (a) In $\mathbb{R}^4$, find the angle between $\langle e_1, e_2, e_3, e_4\rangle$ and $\langle a_1, a_2, a_3, a_4\rangle$, where $a_k = \sum_{i=1}^{k} e_i$.

Solution: Let $P = \langle e_1, e_2, e_3, e_3\rangle$ and $Q = \langle a_1, a_2, a_3, a_4\rangle$. Note that $P = e_1 + U$ and $Q = e_1 + V$ where

$$\begin{aligned}
U &= \text{Span}\{e_2 - e_1, e_3 - e_1, e_4 - e_1\} = \left\{x \in \mathbb{R}^4 \middle| \sum x_i = 0\right\}, \\
V &= \text{Span}\{a_2 - a_1, a_3 - a_1, a_4 - a_3\} = \text{Span}\{e_2, e_2 + e_3, e_2 + e_3 + e_4\} \\
&= \text{Span}\{e_2, e_3, e_4\} = \left\{x \in \mathbb{R}^4 \middle| x_1 = 0\right\}, \\
W &= U \cap V = \left\{x \in \mathbb{R}^4 \middle| x_1 = 0\,,\ x_2 + x_3 + x_4 = 0\right\} = \text{Span}\{e_3 - e_2\,,\ e_4 - e_3\}, \\
W^\perp &= \left\{x \in \mathbb{R}^4 \middle| x \bullet (e_3 - e_2) = x \bullet (e_4 - e_3) = 0\right\} = \left\{x \in \mathbb{R}^4 \middle| x_2 = x_3 = x_4\right\}, \\
U \cap W^\perp &= \left\{x \in \mathbb{R}^4 \middle| \sum x_i = 0\,,\ x_2 = x_3 = x_4\right\} = \text{Span}\{(-3, 1, 1, 1)^T\}, \\
V \cap W^\perp &= \left\{x \in \mathbb{R}^4 \middle| x_1 = 0\,,\ x_2 = x_3 = x_4\right\} = \text{Span}\{(0, 1, 1, 1)^T\}.
\end{aligned}$$

Let $u = (-3, 1, 1, 1)^T$ and $v = (0, 1, 1, 1)^T$ so that we have $U \cap W^\perp = \text{Span}\{u\}$ and $V \cap W^\perp = \text{Span}\{v\}$. Then

$$\theta(P, Q) = \theta(U, V) = \cos^{-1} \frac{|u \bullet v|}{|u|\,|v|} = \cos^{-1} \tfrac{3}{\sqrt{12}\sqrt{3}} = \cos^{-1} \tfrac{1}{2} = \tfrac{\pi}{3}.$$

(b) In $\mathbb{R}^n$, find the distance h_k from $\langle e_1, e_2, \cdots, e_k\rangle$ to $\langle e_{k+1}, e_{k+2}, \cdots, e_n\rangle$.

Solution: Let $P = \langle e_1, \cdots, e_k\rangle$ and $Q = \langle e_{k+1}, \cdots, e_n\rangle$. Let U and V be the associated vector spaces so we have $P = e_1 + U$ and $Q = e_n + V$ where $U = \text{Span}\{u_2, u_3, \cdots, u_k\}$ with $u_i = e_i - e_1$ for $1 < i \leq k$ and $V = \text{Span}\{v_{k+1}, v_{k+2}, \cdots, v_{n-1}\}$ with $v_j = e_j - e_n$ for $k < j < n$. Then

$$U + V = \text{Span}\{u_i, v_j \big| 1 < i \leq k, k < j < n\}$$

and

$$\begin{aligned}
(U + V)^\perp &= \left\{x \in \mathbb{R}^n \middle| x \bullet u_i = x \bullet v_j = 0 \text{ for } 1 < i \leq k, k < j < n\right\} \\
&= \left\{x \in \mathbb{R}^n \middle| x_i = x_1 \text{ for } 1 < i \leq j \text{ and } x_j = x_n \text{ for } k < j < n\right\} \\
&= \text{Span}\{u, v\}
\end{aligned}$$

where $u = \sum_{i=1}^{k} e_i = (1, \cdots, 1, 0, \cdots, 0)^T$ and $v = \sum_{j=k+1}^{n} e_i = (0, \cdots, 0, 1, \cdots, 1)^T$. To find $\text{Proj}_{(U+V)^\perp}(e_n - e_1)$, we let $A = (u, v) \in M_{n\times 2}(\mathbb{R})$ so that $(U + V)^\perp = \text{ColA}$, then we have

$$\begin{aligned}
\text{Proj}_{(U+V)^\perp}(e_n - e_1) &= A(A^TA)^{-1}A^T(e_n - e_1) = (u\,,\ v)\begin{pmatrix} u \bullet u & u \bullet v \\ v \bullet u & v \bullet v \end{pmatrix}^{-1}\begin{pmatrix} u^T \\ v^T \end{pmatrix}(e_n - e_1) \\
&= (u\,,\ v)\begin{pmatrix} k & 0 \\ 0 & n-k \end{pmatrix}^{-1}\begin{pmatrix} u_n - u_1 \\ v_n - v_1 \end{pmatrix} = (u\,,\ v)\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{n-k} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (u\,,\ v)\begin{pmatrix} -\frac{1}{k} \\ \frac{1}{n-k} \end{pmatrix} \\
&= -\tfrac{1}{k}\,u + \tfrac{1}{n-k}\,v = \left(-\tfrac{1}{k}, \cdots, -\tfrac{1}{k}, \tfrac{1}{n-k}, \cdots, \tfrac{1}{n-k}\right)^T.
\end{aligned}$$

Thus

$$h_k = \text{dist}(P, Q) = \left|\text{Proj}_{(U+V)^\perp}(e_n - e_1)\right| = \sqrt{k \cdot \tfrac{1}{k^2} + (n-k) \cdot \tfrac{1}{(n-k)^2}} = \sqrt{\tfrac{1}{k} + \tfrac{1}{n-k}} = \sqrt{\tfrac{n}{k(n-k)}}\,.$$

3: Let U and V be subspaces of $\mathbb{R}^n$.

(a) Show that $(U\cap V)^\perp = U^\perp + V^\perp$.

Solution: We shall show that $U\cap V = (U^\perp + V^\perp)^\perp$. It then follows that $(U\cap V)^\perp = (U^\perp + V^\perp)^{\perp\perp} = U^\perp + V^\perp$.

Let $x\in U\cap V$. Let $y\in U^\perp + V^\perp$, say $y = y_1 + y_2$ with $y_1\in U^\perp$ and $y_2\in V^\perp$. Since $x\in U$ and $y_1\in U^\perp$ we have $x\cdot y_1 = 0$. Since $x\in V$ and $y_2\in V^\perp$ we have $x\cdot y_2 = 0$. Thus $x\cdot y = x\cdot y_1 + x\cdot y_2 = 0$. Since $x\cdot y = 0$ for all $y\in U^\perp + V^\perp$, we have $x\in (U^\perp + V^\perp)^\perp$. This proves that $U\cap V\subseteq (U^\perp + V^\perp)^\perp$.

Now let $x\in (U^\perp + V^\perp)^\perp$ so that $x\cdot y = 0$ for all $y\in U^\perp + V^\perp$. For all $y\in U^\perp$, we also have $y\in U^\perp + V^\perp$, and so $x\cdot y = 0$. Since $x\cdot y = 0$ for all $y\in U^\perp$ we have $x\in U^{\perp\perp} = U$. Similarly we have $x\in V$ so that $x\in U\cap V$. This proves that $(U^\perp + V^\perp)^\perp = U\cap V$, and so we have $U\cap V = (U^\perp + V^\perp)^\perp$, as required.

(b) Show that $\theta(U,V) = 0 \iff \theta(U^\perp, V^\perp) = 0$.

Solution: We claim that $\theta(U,V) = 0$ if and only if either $U\subseteq V$ or $V\subseteq U$. By definition, we know that if either $U\subseteq V$ or $V\subseteq U$ then $\theta(U,V) = 0$. Suppose $U\not\subseteq V$ and $V\not\subseteq U$. Let $W = U\cap V$ and choose $0\neq u\in U\cap W^\perp$ and $0\neq v\in V\cap W^\perp$ so that $\theta(u,v) = \theta(U,V)$. If we had $\theta(u,v) = 0$ then we would have $u = tv$ for some $0\neq t\in\mathbb{R}$ so that $u\in V$, but then we would have $u\in U\cap V = W$ and $u\in W^\perp$ so that $u = 0$. Thus we must have $\theta(u,v)\neq 0$ so $\theta(U,V)\neq 0$. This proves the claim.

Next we claim that $U\subseteq V \iff V^\perp\subseteq U^\perp$. Suppose that $U\subseteq V$. Let $x\in V^\perp$. Let $y\in U$. Since $y\in U$ and $U\subseteq V$ we have $y\in V$. Since $x\in V^\perp$ and $y\in V$ we have $x\cdot y = 0$. Since $x\cdot y$ for all $y\in U$ we have $x\in U^\perp$. This proves that if $U\subseteq V$ then $V^\perp\subseteq U^\perp$. Conversely, if $V^\perp\subseteq U^\perp$ then we have $U^{\perp\perp}\subseteq V^{\perp\perp}$, and so $U\subseteq V$. This proves the claim.

From the above two claims, we see that

$$\theta(U,V) = 0 \iff \big(U\subseteq V \text{ or } V\subseteq U\big) \iff \big(V^\perp\subseteq U^\perp \text{ or } U^\perp\subseteq V^\perp\big) \iff \theta(U^\perp, V^\perp) = 0.$$

(c) Show that $\theta(U,V) = \frac{\pi}{2} \iff \theta(U^\perp, V^\perp) = \frac{\pi}{2}$.

Solution: Let $\mathcal{A} = \{u_1,\cdots,u_k\}$ be a basis for $U\cap W^\perp$, let $\mathcal{B} = \{v_1,\cdots,v_l\}$ be a basis for $V\cap W^\perp$, Let $\mathcal{C} = \{w_1,\cdots,w_m\}$ be a basis for $W = U\cap V$, and let $\mathcal{D} = \{z_1,\cdots,z_p\}$ be a basis for $Z = U^\perp\cap V^\perp = (U+V)^\perp$. Note that $U = W\oplus(U\cap W^\perp)$ because given $u\in U$ we can write u uniquely in the form $u = w + y$ with $w\in W$ and $y\in W^\perp$, and then, since $u\in U$ and $w\in W = U\cap V\subseteq U$, we also have $y\in U$ so that $y\in U\cap W^\perp$. Similarly we have $V = W\oplus(V\cap W^\perp)$. Thus $\mathcal{A}\cup\mathcal{C}$ is a basis for U and $\mathcal{B}\cup\mathcal{C}$ is a basis for V and $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ is a basis for $U+V$ and $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}$ is a basis for all of $\mathbb{R}^n$, and we have $k+l+m+p = n$. In this situation, we write $\mathbb{R}^n = (U\cap W^\perp)\oplus(V\cap W^\perp)\oplus W\oplus Z$. Interchanging U with $V^\perp$ and V with $U^\perp$ and $W = U\cap V$ with $Z = U^\perp\cap V^\perp$, we also have $\mathbb{R}^n = (V^\perp\cap Z^\perp)\oplus(U^\perp\cap Z^\perp)\oplus Z\oplus W$.

Suppose that $\theta(U,V) = \frac{\pi}{2}$. Note that, by Part (b), we do not have $U\subseteq V$ or $V\subseteq U$, and so

$$\tfrac{\pi}{2} = \theta(U,V) = \min\Big\{\theta(u,v)\Big|0\neq u\in U\cap W^\perp\,,\,0\neq v\in V\cap W^\perp\Big\}.$$

It follows that $\theta(u,v) = \frac{\pi}{2}$ for all $0\neq u\in U\cap W^\perp\,,\,0\neq v\in V\cap W^\perp$ (because if we had $\theta(u,v) > \frac{\pi}{2}$ for any such pair, then we would also have $\theta(u,-v) = \pi - \theta(u,v) < \frac{\pi}{2}$). Thus we have $u\cdot v = 0$ for all $u\in U\cap W^\perp\,,\,v\in V\cap W^\perp$.

We claim that $U\cap W^\perp\subseteq V^\perp\cap Z^\perp$ and $V\cap W^\perp\subseteq U^\perp\cap Z^\perp$. Let $u\in U\cap W^\perp$. Since $u\in U$ and $U\subset U+V = Z^\perp$, we also have $u\in Z^\perp$. It remains to show that $u\in V^\perp$. Let $y\in V^\perp$. Write $y = w + v$ with $w\in W$ and $v\in W^\perp$. Since $y\in V$ and $w\in W = U\cap V\subseteq V$ we also have $v\in V$ and so $v\in V\cap W^\perp$. Since $u\in U\cap U^\perp$ and $v\in V\cap W^\perp$, we have $u\cdot v = 0$ (as shown in the previous paragraph). Since $u\in W^\perp$ and $w\in W$ we also have $u\cdot w = 0$. Since $u\cdot w = u\cdot v = 0$ we have $u\cdot y = u\cdot(w+v) = 0$. Since $u\cdot y = 0$ for all $y\in V$ we have $u\in V^\perp$, as required. This proves that $U\cap W^\perp\subseteq V^\perp\cap Z^\perp$. The proof that $V\cap W^\perp\subseteq U^\perp\cap Z^\perp$ is similar.

Since $U\cap W^\perp\subseteq V^\perp\cap Z^\perp$ and $V\cap W^\perp\subseteq U^\perp\cap Z^\perp$, and since

$$\dim(U\cap W^\perp) + \dim(V\cap W^\perp) = n - m - p = \dim(V^\perp\cap Z^\perp) + \dim(U^\perp\cap Z^\perp),$$

it follows that $\dim(U\cap W^\perp) = \dim(V^\perp\cap Z^\perp)$ and $\dim(V\cap W^\perp) = \dim(U^\perp\cap Z^\perp)$, so in fact we see that $U\cap W^\perp = V^\perp\cap Z^\perp$ and $V\cap W^\perp = U^\perp\cap Z^\perp$. Thus

$$\theta(U^\perp, V^\perp) = \min\Big\{\theta(u,v)\Big|0\neq u\in U^\perp\cap Z^\perp\,,\,0\neq v\in V^\perp\cap Z^\perp\Big\} = \theta(U,V) = \tfrac{\pi}{2}.$$

Conversely, if $\theta(U^\perp, V^\perp) = \frac{\pi}{2}$ then $\theta(U,V) = \theta(U^{\perp\perp}, V^{\perp\perp}) = \theta(U^\perp, V^\perp) = \frac{\pi}{2}$.

4: Let U and V be subspaces of $\mathbb{R}^n$ with $0<\theta(U,V)<\frac{\pi}{2}$. Let $W=U\cap V$ and $Z=U^\perp\cap V^\perp$ so that we have

$$\theta(U,V)=\min\left\{\theta(u,v)\Big|0\neq u\in U\cap W^\perp\,,\,0\neq v\in V\cap W^\perp\right\}\text{ , and}$$
$$\theta(U^\perp,V^\perp)=\min\left\{\theta(x,y)\Big|0\neq x\in U^\perp\cap Z^\perp\,,\,0\neq y\in V^\perp\cap Z^\perp\right\}.$$

Choose $u\in U\cap W^\perp$ and $v\in V\cap W^\perp$ with $|u|=|v|=1$ such that $\theta(u,v)=\theta(U,V)$.

(a) Show that $\text{Proj}_V(u)=\text{Proj}_v(u)=(u\cdot v)\,v$ and $\text{Proj}_U(v)=\text{Proj}_u(v)=(u\cdot v)\,u$.

Solution: We already know that $\text{Proj}_v(u)=\frac{u\cdot v}{|v|^2}v=(u\cdot v)v$. We also remark that $0<\theta(u,v)<\frac{\pi}{2}$ (since if $\theta(u,v)>\frac{\pi}{2}$ then we would have $\theta(u,-v)=\pi-\theta(u,v)<\frac{\pi}{2}$ contradicting the minimality of $\theta(u,v)$) and hence $u\cdot v=\cos\theta(u,v)>0$.

Write $u=x+y$ with $x\in V$ and $y\in V^\perp$ so that $x=\text{Proj}_V(u)$. Since $\theta(u,v)\neq\frac{\pi}{2}$ we have $u\cdot v\neq 0$ so $u\notin V^\perp$ hence $u\neq y$ and so $x\neq 0$. Since $W=U\cap V\subseteq V$ we have $V^\perp\subseteq W^\perp$. Since $u\in W^\perp$ and $y\in V^\perp\subseteq W^\perp$, we also have $x=u-y\in W^\perp$. Thus we have $0\neq x\in V\cap W^\perp$. Suppose, for a contradiction, that $x=\text{Proj}_V(u)\neq\text{Proj}_v(u)=(u\cdot v)v$. Then by Trigonometric Ratios and Scaling, and using the fact that $\text{Proj}_V(u)$ is the unique point in V nearest to u, we have

$$\sin\theta(u,x)=|u-x|=\big|u-\text{Proj}_V(u)\big|<\big|u-(u\cdot v)v\big|=\sin\theta\big(u,(u\cdot v)v\big)=\sin\theta(u,v)$$

which implies that $\theta(u,x)<\theta(u,v)$, contradicting the minimality of $\theta(u,v)$.

(b) Let $x=(u\cdot v)u-v$ and $y=u-(u\cdot v)v$. Show that $0\neq x\in U^\perp\cap Z^\perp$, $0\neq y\in V^\perp\cap Z^\perp$ and $\theta(x,y)=\theta(u,v)$.

Solution: By Part (a), we have $y=u-(u\cdot v)v=u-\text{Proj}_V(u)=\text{Proj}_{V^\perp}(u)$. Let $z=\text{Proj}_V(u)=(u\cdot v)v$. Then $u=z+y$ with $z\in V$ and $y\in V^\perp$. Since $u\in U$ and $z\in V$ we have $y=u-z\in U+V=(U^\perp\cap V^\perp)^\perp=Z^\perp$, and so $y\in V^\perp\cap\mathbb{Z}^\perp$. Also note that since $0\neq u\in U\cap W^\perp$ it follows that $u\notin V$ (if we had $u\in V$ then we would have $u\in U\cap V=W$ hence $u\in W\cap W^\perp$ so that $u=0$). Since $u\notin V$ we have $u\neq z$ so that $y\neq 0$. This completes the proof that $0\neq y\in V^\perp\cap Z^\perp$. A similar proof shows that $x=-\text{Proj}_{U^\perp}(v)$ and that $0\neq x\in U^\perp\cap Z^\perp$.

By Trigonometric Ratios and Scaling we have

$$\sin\theta(u,v)=\sin\theta\big(u,(u\cdot v)v\big)=\big|u-(u\cdot v)v\big|=|y|\text{ , and}$$
$$\sin\theta(u,v)=\sin\theta\big((u\cdot v)u,v\big)=\big|v-(u\cdot v)u\big|=|x|,$$

and so

$$\cos\theta(x,y)=\frac{x\cdot y}{|x||y|}=\frac{\big((u\cdot v)u-v\big)\cdot\big(u-(u\cdot v)v\big)}{\sin^2\theta(u,v)}=\frac{(u\cdot v)-(u\cdot v)^3-(u\cdot v)+(u\cdot v)}{\sin^2\theta(u,v)}$$
$$=\frac{(u\cdot v)-(u\cdot v)^3}{\sin^2\theta(u,v)}=\frac{\cos\theta(u,v)-\cos^3\theta(u,v)}{\sin^2\theta(u,v)}=\cos\theta(u,v)$$

and hence $\theta(x,y)=\theta(u,v)$, as required.

(c) Show that $\theta(U^\perp,V^\perp)=\theta(U,V)$.

Solution: Since we can choose $0\neq x\in U^\perp\cap Z^\perp$ and $0\neq y\in V^\perp\cap Z^\perp$ such that $\theta(x,y)=\theta(u,v)=\theta(U,V)$, it follows from the definition of $\theta(U^\perp,V^\perp)$ that $\theta(U^\perp,V^\perp)\leq\theta(x,y)=\theta(u,v)=\theta(U,V)$. By interchanging U with $U^\perp$ and V with $V^\perp$, we also have $\theta(U,V)=\theta(U^{\perp\perp},V^{\perp\perp})\leq\theta(U^\perp,V^\perp)$.