MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 2

1: (a) Let $u_1=(1,0,2,1)^T$, $u_2=(1,1,3,2)^T$, $u_3=(1,-1,2,0)^T$ and $x=(3,2,-1,2)^T$. Let $\mathcal{A}=\{u_1,u_2,u_3\}$ and let $U=\text{Span}\,\mathcal{A}$. Find $\text{Proj}_U(x)$.

Solution: Let $A=\big(u_1,u_2,u_3\big)\in M_{4\times 3}$. We have

$$A=\begin{pmatrix}1&1&1\\0&1&-1\\2&3&2\\1&2&0\end{pmatrix}\ ,\ A^TA=\begin{pmatrix}6&9&5\\9&15&6\\5&6&6\end{pmatrix}\ ,\ A^Tx=\begin{pmatrix}3\\6\\-1\end{pmatrix}$$

and

$$\big(A^TA\big|A^Tx\big)=\left(\begin{array}{ccc|c}6&9&5&3\\9&15&6&6\\5&6&6&-1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&\frac{3}{2}&\frac{5}{6}&\frac{1}{2}\\0&\frac{3}{2}&-\frac{3}{2}&\frac{3}{2}\\0&-\frac{3}{2}&\frac{11}{6}&-\frac{7}{2}\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&\frac{7}{3}&-1\\0&1&-1&1\\0&1&\frac{1}{3}&2\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&13\\0&1&0&-5\\0&0&1&-6\end{array}\right)$$

and so, using Theorem 2.15, we take

$$t=\begin{pmatrix}13\\-5\\-6\end{pmatrix}\quad\text{and}\quad\text{Proj}_U(x)=At=\begin{pmatrix}2\\1\\-1\\3\end{pmatrix}.$$

(b) Let $A=\begin{pmatrix}1&2&1&3&4\\2&3&1&5&6\\1&1&0&1&3\\-1&1&2&2&0\end{pmatrix}\in M_{4\times5}(\mathbb{R})$ and $x=\begin{pmatrix}2\\1\\4\\1\\2\end{pmatrix}\in\mathbb{R}^5$. Find $\text{Proj}_{\text{Null}(A)}(x)$.

Solution: First we find a basis for $\text{Null}(A)$. We have

$$A=\begin{pmatrix}1&2&1&3&4\\2&3&1&5&6\\1&1&0&1&1\\-1&1&2&2&0\end{pmatrix}\sim\begin{pmatrix}1&2&1&3&4\\0&1&1&1&2\\0&1&1&2&1\\0&3&3&5&4\end{pmatrix}\sim\begin{pmatrix}1&0&-1&1&0\\0&1&1&1&2\\0&0&0&1&-1\\0&0&0&2&-2\end{pmatrix}\sim\begin{pmatrix}1&0&-1&0&1\\0&1&1&0&3\\0&0&0&1&-1\\0&0&0&0&0\end{pmatrix}$$

so $\text{Null}(A)$ has basis $\{u,v\}$ where $u=(1,-1,1,0,0)^T$ and $v=(-1,-3,0,1,1)^T$ and we have $\text{Null}(A)=\text{Col(B)}$ where $B=(u,v)\in M_{5\times2}(\mathbb{R})$. Note that

$$B^TB=\begin{pmatrix}|u|^2&u\bullet v\\u\bullet v&|v|^2\end{pmatrix}=\begin{pmatrix}3&2\\2&12\end{pmatrix}\ ,\text{ and}$$

$$B^Tx=\begin{pmatrix}1&-1&1&0&0\\-1&-3&0&1&1\end{pmatrix}\begin{pmatrix}2\\1\\4\\1\\2\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}$$

and so we have

$$\text{Proj}_{\text{Null}(A)}(x)=\text{Proj}_{\text{Col(B)}}(x)=B(B^TB)^{-1}B^Tx=B\cdot\tfrac{1}{32}\begin{pmatrix}12&-2\\-2&3\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix}$$

$$=\tfrac{1}{32}B\begin{pmatrix}64\\-16\end{pmatrix}=\tfrac{1}{2}B\begin{pmatrix}4\\-1\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}1&-1\\-1&-3\\1&0\\0&-1\\0&-1\end{pmatrix}\begin{pmatrix}4\\-1\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}5\\-1\\4\\-1\\-1\end{pmatrix}.$$

2: (a) Let $A \in M_{k\times n}(\mathbb{R})$ with $\text{rank}(A) = k$, and let $b \in \mathbb{R}^n$. Find a formula, in terms of A and b, for the point $x \in \mathbb{R}^n$ with $Ax = Ab$ which is nearest to the origin.

Solution: We have $Ax = Ab \iff A(x-b) = 0 \iff x - b \in \text{Null}A \iff x \in b + \text{Null}A$. For $x = b + u$ with $u \in \text{Null}A$, we have $\text{dist}(x, 0) = |x| = |b + u| = \big|u - (-b)\big|$ and so to minimize $\text{dist}(x, 0)$ we must choose

$$u = \text{Proj}_{\text{Null}A}(-b) = -b - \text{Proj}_{(\text{Null}A)^\perp}(-b) = -b + \text{Proj}_{\text{Col}(A^T)}(b) = -b + A^T(AA^T)^{-1}Ab.$$

Thus the required point x is given by

$$x = b + u = A^T(AA^T)^{-1}A\,b.$$

(b) Find the point $x \in \mathbb{R}^4$, of minimum possible norm, such that $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Solution: First we solve $Ax = b$. We have

$$\big(A\big|b\big) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 3 \\ 2 & 3 & 1 & -1 & 4 \\ 1 & 0 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 2 & 2 & -2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and so the solution set is the affine space $P = p + \text{ColB}$ where

$$p = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The point $x \in p + \text{ColB}$ of minimum norm is the point $x = \text{Proj}_P(0) = p + \text{Proj}_{\text{ColB}}(-p)$. To find this projection, we solve $B^TB\,t = B^T(-p)$ for $t \in \mathbb{R}^2$ and then set $x = p + Bt$. We have

$$B^TB = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \; B^T(-p) = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$\Big(B^TB\Big|B^T(-p)\Big) = \left(\begin{array}{cc|c} 3 & -2 & 3 \\ -2 & 3 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\frac{2}{3} & 1 \\ 0 & \frac{5}{3} & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right)$$

so we take $t = \left(\frac{3}{5}, -\frac{3}{5}\right)^T$ and then the solution x of minimum norm is

$$x = p + Bt = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{pmatrix} = \tfrac{1}{5}\left(\begin{pmatrix} -5 \\ 10 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -6 \\ 3 \\ -3 \end{pmatrix}\right) = \tfrac{1}{5}\begin{pmatrix} 1 \\ 4 \\ 3 \\ -3 \end{pmatrix}.$$

3: (a) Let $0 \neq u, v, w \in \mathbb{R}^n$. Suppose that $w = su + tv$ with $s, t \geq 0$. Show that $\theta(u, v) = \theta(u, w) + \theta(w, v)$.

Solution: First note that by Scaling, we may assume that each of the vectors u, v and w is a unit vector. In this case, note that

$$1 = |w|^2 = (su + tv) \bullet (su + tv) = s^2 + 2st(u \bullet v) + t^2.$$

We have

$$\begin{aligned}
\cos\theta(u, w) &= u \bullet w = u \bullet (su + tv) = s + t(u \bullet v) \\
\sin\theta(u, w) &= \sqrt{1 - \cos^2\theta(u, w)} = \sqrt{1 - (s + t(u \bullet v))^2} \\
&= \sqrt{(s^2 + 2st(u \bullet v) + t^2) - (s^2 + 2st(u \bullet v) + t^2(u \bullet v)^2} \\
&= \sqrt{t^2 - t^2(u \bullet v)^2} = t\sqrt{1 - (u \bullet v)^2}
\end{aligned}$$

and similarly

$$\begin{aligned}
\cos\theta(v, w) &= t + s(u \bullet v) \\
\sin\theta(v, w) &= s\sqrt{1 - (u \bullet v)^2}
\end{aligned}$$

and so

$$\begin{aligned}
\cos\big(\theta(u, w) + \theta(v, w)\big) &= (s + t(u \bullet v))(t + s(u \bullet v)) - t\sqrt{1 - (u \bullet v)^2} \cdot s\sqrt{1 - (u \bullet v)^2} \\
&= \big(st + s^2(u \bullet v) + t^2(u \bullet v) + st(u \bullet v)^2\big) - st\big(1 - (u \bullet v)^2\big) \\
&= s^2(u \bullet v) + t^2(u \bullet v) + 2st(u \bullet v)^2 \\
&= \big(s^2 + t^2 + 2st(u \bullet v)\big)(u \bullet v) \\
&= u \bullet v = \cos\theta(u, v) \text{ , and} \\
\sin\big(\theta(u, w) + \theta(v, w)\big) &= \sin\theta(u, w)\cos\theta(v, w) + \cos\theta(u, w)\sin\theta(v, w) \\
&= t\sqrt{1 - (u \bullet v)^2}\big(t + s(u \bullet v)\big) + \big(s + t(u \bullet v)\big) \cdot s\sqrt{1 - (u \bullet v)^2} \\
&= \big(t^2 + st(u \bullet v) + s^2 + st(u \bullet v)\big)\sqrt{1 - (u \bullet v)^2} \\
&= \sqrt{1 - (u \bullet v)^2} = \sin\theta(u, v).
\end{aligned}$$

(b) Let $[a, b, c]$ be a triangle in $\mathbb{R}^n$. Let $\alpha = \angle bac$, $\beta = \angle cba$ and $\gamma = \angle acb$. Show that $\alpha + \beta + \gamma = \pi$.

Solution: Let $u = c - b$, $v = a - c$ and $w = b - a$ so that $\alpha = \theta(w, -v)$, $\beta = \theta(u, -w)$ ad $\gamma = \theta(v, -u)$. Note that $u + v + w = (c - b) + (a - c) + (b - a) = 0$ so that $-w = u + v$, and so $\theta(u, -w) + \theta(v, -w) = \theta(u, v)$ by Part (a). By Symmetry and Scaling, we also have $\theta(w, -v) = \theta(v, -w)$ and $\theta(v, -u) = \pi - \theta(u, v)$, and so

$$\begin{aligned}
\alpha + \beta + \gamma &= \theta(w, -v) + \theta(u, -w) + \theta(v, -u) \\
&= \theta(v, -w) + \theta(u, -w) + \pi - \theta(u, v) \\
&= \theta(u, v) + \pi - \theta(u, v) = \pi.
\end{aligned}$$