MATH 245 Linear Algebra 2, Solutions to the Exercise for Chapter 1

1: Let $P = a + \text{Null}(A)$ and $Q = b + \text{Col(B)}$ where

$$a = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 3 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 3 & -1 \end{pmatrix}.$$

Find a point $p \in \mathbb{R}^4$ and a basis for a subspace $U \subseteq \mathbb{R}^4$ such that $P \cap Q = p + U$.

Solution: Let $x \in P \cap Q$. Choose $u \in \text{Null}A$ and $v \in \text{ColB}$ so that $x = a + u = b + v$. Since $v \in \text{ColB}$ we can choose $t \in \mathbb{R}^3$ so that $v = Bt$. Then we have $a + u = b + Bt$. Multiply (on the left) by A, using the fact that $Au = 0$, to get $Aa + Ab + ABt$. Thus we must have $ABt = A(a - b)$. Conversely, suppose that $t \in \mathbb{R}^3$ satisfies $ABt = A(a - b)$ and let $x = b + Bt$. Then $x \in b + \text{ColB} = \text{Q}$. Also, we have $A(x - a) = A(b + Bt - a) = ABt - A(a - b) = 0$ so that $x - a \in \text{Null}A$, and so $x \in a + \text{Null}A = P$. Thus we have shown that $P \cap Q$ is equal to the set of vectors $x \in \mathbb{R}^4$ with $x = b + Bt$ for some $t \in \mathbb{R}^3$ such that $ABt = A(a - b)$. We have

$$AB = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 3 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 14 & 6 \\ 3 & 8 & 3 \end{pmatrix},$$

$$A(a-b) = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 3 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix}, \text{ and}$$

$$\Big(AB \Big| A(a-b)\Big) = \left(\begin{array}{ccc|c} 0 & -2 & 0 & 2 \\ 6 & 14 & 6 & 10 \\ 3 & 8 & 3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 8 & 3 & 4 \\ 6 & 14 & 6 & 10 \\ 0 & -2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{8}{3} & 1 & \frac{4}{3} \\ 0 & 2 & 0 & -2 \\ 0 & -2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so the solution to $ABt = A(a-b)$ is given by $t = c + sv$ where $c = (4, -1, 0)^T$ and $v = (-1, 0, 1)^T$. Thus $P \cap Q$ is the set of points x of the form $x = b + Bt = b + B(c + sv) = b + Bc + sBv$ and so $P \cap Q = p + U$ where

$$p = b + Bc = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

and $U = \text{Span}\{u\}$ with

$$u = Bv = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}.$$

2: (a) Show that the set $A = \{(x,y) \in \mathbb{R}^2 \big| x^2 + y^2 \geq (x^2+y^2-x)^2\}$ is not convex.

Solution: For $(x,y) = \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$ we have $x^2+y^2 = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$ and $(x^2+y^2-x)^2 = (\frac{1}{4}+\frac{1}{4})^2 = \frac{1}{4}$ and so $(x,y) \in A$. For $(x,y) = \left(-\frac{1}{4}, 0\right)$ we have $x^2+y^2 = \frac{1}{16}$ and $(x^2+y^2-x)^2 = \left(\frac{1}{16}+\frac{1}{4}\right)^2 = \left(\frac{5}{16}\right)^2 = \frac{25}{16}\cdot\frac{1}{16} > \frac{1}{16}$ and so $(x,y) \notin A$. Since $a = \left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right) \in A$ and $b = \left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right) \in A$ but $\frac{1}{2}a + \frac{1}{2}b = \left(-\frac{1}{4}, 0\right) \notin A$, we see that A is not convex. We remark that the motivation for selecting the above points a and b comes from recognizing that A is the cardioid given in polar coordinates by $r = 1 + \cos\theta$.

(b) Show that the set $B = \{(x,y) \in \mathbb{R}^2 \big| y \geq x^2\}$ is convex.

Solution: Let $(a,b) \in B$ and let $(c,d) \in B$ and note that $b \geq a^2$ and $d \geq c^2$. Let $(x,y) \in \big[(a,b),(c,d)\big]$, say $(x,y) = (1-t)(a,b) + t(c,d) = \big((1-t)a + tc\,,\,(1-t)b + td\big)$ with $0 \leq t \leq 1$. Then

$$\begin{aligned}
y - x^2 &= \big((1-t)b + td\big) - \big((1-t)a + tc\big)^2 \\
&\geq \big((1-t)a^2 + tc^2\big) - \big((1-t)a + tc\big)^2 \\
&= \big((1-t)a^2 + tc^2\big) - \big((1-t)^2a^2 + 2t(1-t)ac + t^2c^2\big) \\
&= \big((1-t) - (1-t)^2\big)a^2 - 2t(1-t)ac + \big(t - t^2\big)c^2 \\
&= t(1-t)a^2 - 2t(1-t)ac + t(1-t)c^2 \\
&= t(1-t)(a-c)^2 \geq 0
\end{aligned}$$

and so $(x,y) \in B$. Thus B is convex.

3: Let W be a vector space over $\mathbb{R}$. A nonempty set $\emptyset \neq C \subseteq W$ is called **conical** when it has the property that

$$\text{for all } a \in C \text{ and all } 0 \leq t \in \mathbb{R}, \text{ we have } ta \in C.$$

(a) Show that the intersection of a set of conical sets in W is conical.

Solution: Before we begin our solution, we remark that for every convex conical set $\emptyset \neq S \subseteq \mathbb{R}^n$ we have $0 \in S$ because since $S \neq \emptyset$ we can choose an element $a \in S$, and then since S is conical we have $0 = 0 \cdot a \in S$. Now we begin our solution. Let A be a nonempty set. For each $\alpha \in A$, let $\emptyset \neq S_\alpha \subseteq \mathbb{R}^n$ be a conical set in $\mathbb{R}^n$. Let $S = \bigcap_{\alpha \in A} S_\alpha$. Note that $0 \in S$ (so that $S \neq \emptyset$) since $0 \in S_\alpha$ for all $\alpha \in A$. Let $a \in S$ and let $0 \leq t \in \mathbb{R}$. Then $a \in S_\alpha$ for every α, and so $ta \in S_\alpha$ for every α (since S_α is conical and $t \geq 0$), and so $ta \in S$. Thus S is conical.

(b) For a nonempty set $\emptyset \neq \mathcal{S} \subseteq W$ we define the **convex cone** of $\mathcal{S}$, denoted by $\text{Cone}(\mathcal{S})$, to be the smallest convex conical subset of W which contains $\mathcal{S}$, or equivalently the intersection of all convex conical sets in W which contain $\mathcal{S}$. Show that

$$\text{Cone}(\mathcal{S}) = \left\{ \sum_{i=0}^{n} t_i a_i \middle| n \in \mathbb{N}, a_i \in \mathcal{S}, 0 \leq t_i \in \mathbb{R} \right\}.$$

Solution: Let T be the set on the right. Note that $S \subseteq T$ because when $m = 0$, $a_0 \in S$ and $s_0 = 1$ we have $\sum\limits_{i=0}^{m} s_i a_i = a_0$. Note that T is conical because given $0 \leq t \in \mathbb{R}$ and $a \in T$, say $a = \sum\limits_{i=0}^{m} s_i a_i$ where $m \in \mathbb{N}$, $a_i \in S$ and $0 \leq s_i \in \mathbb{R}$, we have $ta = \sum\limits_{i=0}^{m} (ts_i) a_i$ and $0 \leq ts_i \in \mathbb{R}$. We claim that T is convex. Let $a, b \in T$, say $a = \sum\limits_{i=0}^{m} s_i a_i$ and $b = \sum\limits_{i=0}^{m} t_i a_i$ where $m \in \mathbb{N}$, $a_i \in S$ and $0 \leq s_i, t_i \in \mathbb{R}$. Let $x \in [a, b]$, say $x = a + r(b - a)$ with $0 \leq r \leq 1$. Then $x = \sum s_i a_i + r\big(\sum t_i a_i - \sum s_i a_i\big) = \sum r_i a_i$ where $r_i = s_i + r(t_i - s_i)$. Since $s_i \geq 0$ and $t_i \geq 0$ and $r_i \in [s_i, t_i]$, we have $r_i \geq 0$, and hence $x = \sum r_i a_i \in T$. Thus T is convex, as claimed. Since $S \subseteq T$ and T is convex and conical, we have $\text{Cone}(S) \subseteq T$, by the definition of $\text{Cone}(S)$.

Let $C \subseteq \mathbb{R}^n$ be any convex conical set in $\mathbb{R}^n$ which contains S. Let $x \in T$, say $x = \sum\limits_{i=0}^{m} s_i a_i$ where $m \in \mathbb{N}$, $a_i \in S$ and $0 \leq s_i \in \mathbb{R}$. If $x = 0$ then we have $x \in C$ by our preliminary remark from Part (a). Suppose that $x \neq 0$. Then some $s_i > 0$ and so $\sum s_i > 0$. Let $s = \sum s_i$. Note that $\frac{1}{s} x = \sum \frac{s_i}{s} a_i$ and each $\frac{s_i}{s} \geq 0$ with $\sum \frac{s_i}{s} = 1$, and so we have $\frac{1}{s} x \in [S]$. Since C is convex and $S \subseteq C$ we have $[S] \subseteq C$ and so $\frac{1}{s} x \in C$. Since C is conical, $\frac{1}{s} x \in C$ and $s \geq 0$, we have $x = s \cdot \frac{1}{s} c \in C$. Thus $T \subseteq C$. Since $T \subseteq C$ for every convex conical set $C \subseteq \mathbb{R}^n$ with $S \subseteq C$, it follows, from the definition of the convex cone of S, that $T \subseteq \text{Cone}(S)$.

4: Let V and W be vector spaces over a field F in which $2 \neq 0$. Here is an alternate definition for an affine space in V. Let us say that a nonempty set $\emptyset \neq P \subseteq V$ is **affine** in V when $sx + ty \in P$ for all $x, y \in P$ and for all $s, t \in F$ with $s + t = 1$. Also, let us say that a map $A : V \to W$ is **affine** when

$$A(sx + ty) = s\,A(x) + t\,A(y) \text{ for all } x, y \in V \text{ and } s, t \in F \text{ with } s + t = 1.$$

(a) For $\emptyset \neq P \subseteq V$, show that P is affine if and only if $P = a + U$ for some $a \in V$ and some subspace $U \subseteq V$.

Solution: Suppose first that P is affine. Choose a point $a \in P$ and let $U = P - a = \{x - a | x \in P\}$ so that we have $P = a + U$. We claim that U is a subspace of V. Note that $0 = a - a \in U$. Let $u \in U$, say $u = x - a$ with $x \in P$, and let $t \in F$. Then we have $tu + a = t(x - a) + a = tx + (1 - t)a$ which lies in P since $x \in P$ and $a \in P$ and P is affine. Since $u + a \in P$ we have $u \in P - a = U$, so U is closed under multiplication by a scalar. Now let $u, v \in U$, say $u = x - a$ and $v - y - b$ with $x, y \in P$. Note that $u + v + a = (x - a) + (y - a) + a = x + y - a = 2 \cdot \frac{x+y}{2} - 1 \cdot a$. Since $x \in P$ and $y \in P$ and P is affine, we have $\frac{x+y}{2} \in P$. Since $\frac{x+y}{2} \in P$ and $a \in P$ and P is affine, we have $u + v + a = 2 \cdot \frac{x+y}{2} - 1 \cdot a \in P$. Since $u + v + a \in P$ we have $u + v \in P - a = U$, so U is closed under addition. This completes the proof that U is a vector space, as claimed.

Conversely, suppose that $P = a + U$ where $a \in P$ and $U \subseteq V$ is a subspace. Let $x, y \in P$, say $x = a + u$ and $y = a + v$ with $u, v \in U$, and let $s, t \in F$ with $s + t = 1$. Then we have $sx + ty = s(a + u) + t(a + v) = (s + t)a + (su + tv) = a + (su + tv)$, which lies in $a + U$ since $su + tv \in U$. Thus P is affine.

(b) Show that the affine maps $A : V \to W$ are the maps of the form $A(x) = a + L(x)$ for some point $a \in W$ and some linear map $L : V \to W$.

Solution: Suppose that $A : V \to W$ is affine. Let $a = A(0)$ and define $L : V \to W$ by $L(x) = A(x) - A(0)$ so that we have $A(x) = a + L(x)$ for all $x \in V$. We claim that L is linear. Let $u, v \in V$ and let $t \in F$. Then

$$\begin{aligned} L(u) + L(v) &= A(u) - A(0) + A(v) - A(0) = \Big(2\big(\tfrac{1}{2}A(u) + \tfrac{1}{2}A(v)\big) - A(0)\Big) - A(0) \\ &= \Big(2 \cdot A\big(\tfrac{1}{2}u + \tfrac{1}{2}v\big) - 1 \cdot A(0)\Big) - A(0) = A\big(2 \cdot \big(\tfrac{1}{2}u + \tfrac{1}{2}v\big) - 1 \cdot 0\big) - A(0) \\ &= A(u + v) - A(0) = L(u + v) \end{aligned}$$

and

$$\begin{aligned} t\,L(u) &= t\big(A(u) - A(0)\big) = \big(t\,A(u) + (1 - t)A(0)\big) - A(0) = A\big(t \cdot u + (1 - t) \cdot 0\big) - A(0) \\ &= A(tu) - A(0) = L(u) \end{aligned}$$

and so the map L is linear, as claimed.

Conversely, suppose that $L : V \to W$ is linear, let $a \in W$ and define $A : V \to W$ by $A(x) = a + L(x)$. Then for $x, y \in V$ and $s, t \in \mathbb{R}$ with $s + t = 1$, we have

$$sA(x) + tA(y) = s\big(a + L(x)\big) + t\big(a + L(y)\big) = (s + t)a + sL(x) + tL(y) = a + L(sx + ty) = A(sx + ty)$$

and so A is affine.

(c) Show that when $F = \mathbb{R}$, if $A : V \to W$ is affine and $C \subseteq V$ is convex, then the image $A(C)$ is convex.

Solution: Let $A : V \to W$ be affine. Let $C \subseteq V$ be convex. Let $u, v \in A(C) = \big\{A(x)\big| x \in C\big\}$, say $u = A(a)$ and $v = A(b)$ where $a, b \in C$. Let $y \in [u, v]$, say $y = su + tv$ where $0 \leq s, t \in \mathbb{R}$ with $s + t = 1$. Let $x = sa + tb$. Since $0 \leq s, t$ and $s + t = 1$, we have $x \in [a, b]$. Since $a, b \in C$ and $x \in [a, b]$ and C is convex, we have $x \in C$. Since A is affine and $s + t = 1$, we have $A(x) = A(sa + tb) = sA(a) + tA(b) = su + tv = y$ and so $y \in A(C)$. Thus $[u, v] \subseteq A(C)$, and so $A(C)$ is convex.

5: (a) Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{R}^n$ and let $x \in [\mathcal{S}]$. Show that $x = \sum_{i=0}^{m} t_i a_i$ for some $m \in \mathbb{N}$ with $m \leq n$, some $a_i \in \mathcal{S}$, and some $0 \leq t_i \in \mathbb{R}$ with $\sum t_i = 1$.

Solution: Write x in the form $x = \sum_{i=0}^{m} s_i a_i$ where $m \in \mathbb{N}$, $a_i \in S$ and $0 \leq s_i \in \mathbb{R}$ with $\sum s_i = 1$, with the value of $m \in \mathbb{N}$ chosen to be as small as possible. Note that the points a_i must be distinct, since if we had $a_j = a_k$ with $j \neq k$ then we could replace the two terms $s_j a_j + s_k a_k$ in the sum $\sum s_i a_i$ by the single term $(s_j + s_k)a_k$. Suppose, for a contradiction, that $m > n$. Note that, since $m > n$ and the a_i are distinct, the set $\{a_0, a_1, \cdots, a_m\}$ is affinely dependent (because the set of m distinct vectors $\{a_1 - a_0, a_2 - a_0, \cdots, a_m - a_0\}$ is linearly dependent). Choose coefficients t_i, not all zero, so that $\sum_{i=0}^{m} t_i a_i = 0$ and $\sum_{i=0}^{m} t_i = 0$. Note that at least one the coefficients t_i is positive. Choose an index k so that $t_k > 0$ and $\frac{s_k}{t_k} = \min\left\{\frac{s_i}{t_i}\middle| t_i > 0\right\}$, and let $r = \frac{s_k}{t_k}$. Then we have

$$x = \sum_{i=0}^{m} s_i a_i - r \cdot 0 = \sum_{i=0}^{m} s_i a_i - r \sum_{i=0}^{m} t_i a_i = \sum_{i=0}^{m} r_i a_i$$

where $r_i = s_i - r\, t_i$. By our choice of k we have $r_i \geq 0$ for all i $\big($indeed if $t_i \leq 0$ then $r_i = s_i - \frac{s_k}{t_k} t_i \geq s_i \geq 0$ and if $t_i > 0$ then $\frac{s_k}{t_k} \leq \frac{s_i}{t_i}$ so $r_i = s_i - \frac{s_k}{t_k}\, t_i \geq s_i - \frac{s_i}{t_i}\, t_i = 0\big)$ and we have $r_k = s_k - \frac{s_k}{t_k}\, t_k = 0$. Also note that $\sum r_i = \sum s_i - r \sum t_i = 1 - r \cdot 0 = 1$. Thus we have $x = \sum_{i=0}^{m} r_i a_i = \sum_{i \neq k} r_i a_i$ with each $r_i \geq 0$ and $\sum_{i \neq k} r_i = 1$, contradicting the minimality of m.

(b) Let $\mathcal{S} \subseteq \mathbb{R}^n$ with $|\mathcal{S}| \geq n+2$. Show that there exist disjoint, nonempty subsets $A, B \subseteq \mathcal{S}$ such that $[A] \cap [B] \neq \emptyset$.

Solution: Choose $n+2$ distinct points $a_0, a_1, \cdots, a_{n+1} \in S$. We claim that there exist non-empty disjoint sets of indices $I, J \subseteq \{0, 1, 2, \cdots, n+1\}$ such that $\big[\{a_i | i \in I\}\big] \cap \big[\{a_j | j \in J\}\big] \neq \emptyset$, and so we can take $A = \{a_i | i \in I\}$ and $B = \{a_j | j \in J\}$. Since $\{a_0, a_1, \cdots, a_{n+1}\}$ is affinely dependent, we can choose coefficients t_i, not all zero, so that $\sum_{i=0}^{n+1} t_i a_i = 0$ and $\sum_{i=0}^{n+1} t_i = 0$. Let $I = \{i | t_i > 0\}$ and let $J = \{j | t_j < 0\}$. Note that I and J are both nonempty since the coefficients t_i are not all zero and $\sum t_i = 0$ so that at least one coefficient is positive and at least one is negative. For each $j \in J$, let $s_j = -t_j$. Since $\sum_{i=0}^{n+1} t_i = 0$ we have

$$0 = \sum_{i \in I} t_i + \sum_{j \in J} t_j = \sum_{i \in I} t_i - \sum_{j \in J} s_j\,.$$

so we have $\sum_{i \in I} t_i = \sum_{j \in J} s_j$. Let $r = \sum_{i \in I} t_i = \sum_{j \in J} s_j$. Note that $r > 0$ and we have $\sum_{i \in I} \frac{t_i}{r} = 1$ and $\sum_{j \in J} \frac{s_j}{r} = 1$. Since

$$0 = \sum_{i=0}^{n+1} \frac{t_i}{r}\, a_i = \sum_{i \in I} \frac{t_i}{r}\, a_i - \sum_{j \in J} \frac{s_i}{r}\, a_j$$

we have $\sum_{i \in I} \frac{t_i}{r}\, a_i = \sum_{j \in J} \frac{s_i}{r}\, a_j$. Let $x = \sum_{i \in I} \frac{t_i}{r}\, a_i = \sum_{j \in J} \frac{s_i}{r}\, a_j$. Since $x = \sum_{i \in I} \frac{t_i}{r}\, a_i$ with each $\frac{t_i}{r} > 0$ and $\sum_{i \in I} \frac{t_i}{r} = 1$, we have $x \in \big[\{a_i | i \in I\}\big]$. Since $x = \sum_{j \in J} \frac{s_j}{r}\, a_j$ with each $\frac{s_j}{r} > 0$ and $\sum_{j \in J} \frac{s_j}{r} = 1$, we have $x \in \big[\{a_j | j \in J\}\big]$. Thus $\big[\{a_i | i \in I\}\big] \cap \big[\{a_j | j \in J\}\big] \neq \emptyset$.

6: For $x, y \in \mathbb{R}^n$, write $x \leq y$ when $x_i \leq y_i$ for all i. Let $P = \left\{x \in \mathbb{R}^4 \middle| Ax = a \text{ and } Bx \leq b\right\}$ where

$$a = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \ , \ b = \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} \ , \ A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 3 \\ 1 & 2 & 0 & -1 \\ 2 & 1 & -1 & -2 \end{pmatrix} .$$

Find a set of distinct points $a_0, a_1, a_2, a_3 \in \mathbb{R}^4$ such that $P = [a_0, a_1, a_2, a_3]$.

Solution: First we solve $Ax = a$. We have

$$(A|a) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 2 & 3 & 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -3 & -1 & -1 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right) .$$

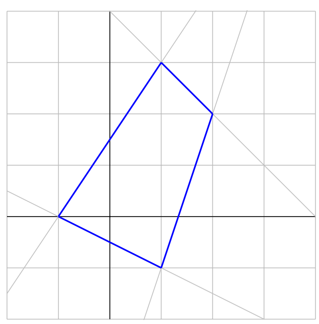
Thus

$$Ax = a \iff x = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = p + t_1 u_1 + t_2 u_2$$

for some $t_1, t_2 \in \mathbb{R}$, where $p = (-1, 2, 0, 0)^T$, $u_1 = (3, -2, 1, 0)^T$ and $u_2 = (1, -1, 0, 1)^T$, and then we have

$$\begin{aligned} Bx \leq b \iff & \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 3 \\ 1 & 2 & 0 & -1 \\ 2 & 1 & -1 & -2 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right) \leq \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} \\ \iff & \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ -3 \\ -1 \\ 3 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \leq \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} \\ \iff & t_1 + t_2 \leq 4 \ , \ -3t_1 + 2t_2 \leq 3 \ , \ -t_1 - 2t_2 \leq 1 \text{ and } 3t_1 - t_2 \leq 4. \end{aligned}$$

The set of solutions (t_1, t_2) is shown below. The lines $t_1 + t_2 = 4$, $-3t_1 + 2t_2 = 3$, $-t_1 - 2t_2 = 1$ and $3t_1 - t_2 = 4$ are shown in grey and the solution set is outlined in blue.



The vertices lie at $(t_1, t_2) = (-1, 0), (1, -1), (2, 2), (1, 3)$ which correspond to the points

$$x = p + t_1 u_1 + t_2 u_2 = \begin{pmatrix} -4 \\ 4 \\ -1 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} , \begin{pmatrix} 7 \\ -4 \\ 2 \\ 2 \end{pmatrix} , \begin{pmatrix} 5 \\ -3 \\ 1 \\ 3 \end{pmatrix} .$$

These 4 points are the required vertices a_0, a_1, a_2, a_3 for which $S = [a_0, a_1, a_2, a_3]$.

We remark that the above solution is not rigorous as it makes use of a picture. To make it rigorous there are several things that should be proven. Indeed, letting $b_0 = (-1, 0)^T$, $b_1 = (1, -1)^T$, $b_2 = (2, 2)^T$ and $b_3 = (1, 3)^T$, and letting $T = \left\{t \in \mathbb{R}^2 \middle| t_1 + t_2 \leq 4, -3t_1 + 2t_2 \leq 3, -t_1 - 2t_2 \leq 1, 3t_1 - t_2 \leq 4\right\}$, one needs to show that $T = [b_0, b_1, b_2, b_3]$ and that the affine map $F(t) = p + t_1 u_1 + t_2 u_2$ sends the convex hull $[b_0, b_1, b_2, b_3]$ to the convex hull $[a_0, a_1, a_2, a_3]$. We do this on the next page.

First we show that the affine map $F:\mathbb{R}^2\to\mathbb{R}^4$ given by $F(t)=p+t_1u_1+t_2u_2$ sendss $[b_0,b_1,b_2,b_3]$ to $[a_0,a_1,a_2,a_3]$. This follows from the following lemma (which some students may have proven as part of their solution to Problem 3(c)).

Lemma: Let V and W be vector spaces over $\mathbb{R}$. Let $F:V\to W$ be an affine map. Let $b_0,\cdots,b_l\in V$. Then F maps $[b_0,\cdots,b_l]$ to $[a_0,\cdots,a_l]$ where $a_i=F(b_i)$.

Proof: We claim that for $n\in\mathbb{N}$, $b_i\in V$ and $0\le s_i\in\mathbb{R}$ with $\sum\limits_{i=0}^{n}s_i=1$ we have $F\big(\sum\limits_{i=0}^{n}s_ib_i\big)=\sum\limits_{i=0}^{n}s_iF(b_i)$. We prove this claim using induction. When $n=0$ the claim holds since $F\big(\sum\limits_{i=0}^{0}s_ib_i\big)=F(1\cdot b_0)=1\cdot F(b_0)$. Suppose the claim holds for some fixed $n\in\mathbb{N}$. Let $b_i\in V$ and $0\le s_i\in\mathbb{R}$ with $\sum\limits_{i=0}^{n+1}s_i=1$. If $s_{n+1}=0$ then we have $F\big(\sum\limits_{i=0}^{n+1}s_ib_i\big)=F\big(\sum\limits_{i=0}^{n}s_ib_i\big)=\sum\limits_{i=0}^{n}s_iF\big(b_i\big)=\sum\limits_{i=0}^{n+1}s_iF\big(b_i\big)$. Suppose that $s_{n+1}\neq 0$. Then

$$\begin{aligned}F\Big(\sum_{i=0}^{n+1}s_ib_i\Big)&=F\Big(\sum_{i=0}^{n}s_ib_i+s_{n+1}b_{n+1}\Big)=F\Big((1-s_{n+1})\sum_{i=0}^{n}\tfrac{s_i}{1-s_{n+1}}\,b_i+s_{n+1}b_{n+1}\Big)\\&=(1-s_{n+1})F\Big(\sum_{i=0}^{n}\tfrac{s_i}{1-s_{n+1}}\,b_i\Big)+s_{n+1}F(b_{n+1})\ \text{, since } F \text{ is affine,}\\&=(1-s_{n+1})\sum_{i=0}^{n}\tfrac{s_i}{1-s_{n+1}}F(b_i)+s_{n+1}F(b_{n+1})\ \text{, by the induction hypothesis,}\\&=\sum_{i=0}^{n+1}s_iF(b_i).\end{aligned}$$

By induction, the claim holds for all $n\in\mathbb{N}$, and the lemma follows from the claim.

Next we show that $[b_0,b_1,b_2,b_3]\subseteq T$. It is easy to check that $b_i\in T$ for all i (simply check that each b_i satisfies the inequalities), so it suffices to show that T is convex. This follows from the following lemma.

Lemma: Let $A\in M_{l\times n}(\mathbb{R})$ and let $b\in\mathbb{R}^l$. Then the set $T=\big\{x\in\mathbb{R}^n\big|Ax\le b\big\}$ is convex.
Proof: Let $u,v\in T$ so we have $Au\le b$ and $Av\le b$, which means that $(Au)_i\le b_i$ and $(Av)_i\le b_i$ for all indices i. Let $x\in[u,v]$, say $x=su+tv$ with $0\le s,t$ and $s+t=1$. Then $Ax=A(su+tv)=sAu+tAv$, so the i^{th} entry of Ax is

$$(Ax)_i=s(Au)_i+t(Av)_i\in\big[(Au)_i,(Av)_i\big].$$

Since $(Au)_i\le b_i$ and $(Av)_i\le b_i$ and $(Ax)_i\in\big[(Au)_i,(Av)_i\big]$, it follows that $(Ax)_i\le b_i$. Since $(Ax)_i\le b_i$ for all i, we have $Ax\le b$ so that $x\in T$. Thus T is convex.

It remains to show that $T\subseteq[b_0,b_1,b_2,b_3]$. We sketch a proof. Let $t\in T$, so we have $t_1+t_2\le 4$, $-3t_1+2t_2\le 3$, $-t_1-2t_2\le 1$ and $3t_1-t_2\le 4$. We consider the two cases that $t_1\le 1$ and $t_1\ge 1$. Suppose first that $t_1\le 1$. From the picture, it appears that $t\in[b_0,b_1,b_3]$, so we solve the system $\sum\limits_{i=0}^{3}s_ib_i=t$, $\sum\limits_{i=0}^{3}s_i=1$ and $s_2=0$ for $s\in\mathbb{R}^4$ to get $s_0=\frac12-\frac12t_1$, $s_1=\frac38+\frac38t_1-\frac14t_2$, $s_2=0$ and $s_3=\frac18+\frac18t_1+\frac14t_2$. We note that each $s_i\ge 0$ (indeed $s_0=\frac12-\frac12t_1\ge 0$ since $t_1\le 1$, $s_1=\frac38-\frac18(-3t_1+2t_2)\ge 0$ since $-3t_1+2t_2\le 3$ and $s_3=\frac18-\frac18(-t_1-2t_2)\ge 0$ since $-t_1-2t_2\le 1$) and also that $\sum s_i=1$ and so we have $t=\sum s_ib_i\in[b_0,b_1,b_2,b_3]$. Next we suppose that $t_1\ge 1$. From the picture, it appears that $t\in[b_1,b_2,b_3]$, so we solve $\sum\limits_{i=0}^{3}s_ib_i=t$, $\sum\limits_{i=0}^{3}s_i=1$ and $s_0=0$ for $s\in\mathbb{R}^4$ to get $s_0=0$, $s_1=1-\frac14t_1-\frac14t_2$, $s_2=-1+t_1$ and $s_3=1-\frac34t_1+\frac14t_2$. We note that each $s_i\ge 0$ (indeed $s_1=1-\frac14(t_1+t_2)\ge 0$ since $t_1+t_2\le 4$, $s_2=-1+t_1\ge 0$ since $t_1\ge 1$ and $s_3=1-\frac14(3t_1-t_2)\ge 0$ since $3t_1-t_2\le 4$) and also that $\sum s_i=1$ and so we have $t=\sum s_ib_i\in[b_0,b_1,b_2,b_3]$. In either case we find that $t\in[b_0,b_1,b_2,b_3]$, so we have $T\subseteq[b_0,b_1,b_2,b_3]$.