MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 12

1: Find the number of similarity classes of 8×8 complex matrices whose distinct eigenvalues are 1 and 2.

Solution: Recall that two 8×8 matrices are similar if and only if they have the same Jordan form (up to the order of the Jordan blocks). For an 8×8 matrix A with eigenvalues $\lambda_1=1$ and $\lambda_2=2$, let $k_1=\dim(K_1)$ and $k_2=\dim(K_2)$. Note that $k_1+k_2=8$ so the only possibilities for the pair (k_1,k_2) are

$$(k_1,k_2)=(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1)\,.$$

For each possible value of $k=k_1$ or k_2, we list all the possibilities for the sizes of the Jordan blocks for $\lambda=\lambda_1$ or λ_2; here the ordered l-tuple $(m_1,\cdots,m_l)$ indicates l blocks of sizes $m_1,\cdots,m_l$ with $\sum m_i=k$:

k	possible block sizes
1	(1)
2	$(1,1),(2)$
3	$(1,1,1),(2,1),(3)$
4	$(1,1,1,1),(2,1,1),(2,2),(3,1),(4)$
5	$(1,1,1,1,1),(2,1,1,1),(2,2,1),(3,1,1),(3,2),(4,1),(5)$
6	$(1,1,1,1,1,1),(2,1,1,1,1),(2,2,1,1),(2,2,2),(3,1,1,1),(3,2,1),(3,3),(4,1,1),(4,2),(5,1),(6)$
7	$(1,1,1,1,1,1,1),(2,1,1,1,1,1),(2,2,1,1,1),(2,2,2,1),(3,1,1,1,1),(3,2,1,1),(3,2,2),(3,3,1),$ $(4,1,1,1),(4,2,1),(4,3),(5,1,1),(5,2),(6,1),(7)$

Thus for $k=1,2,\cdots,7$, we obtain the following number $p(k)$ of possible block sizes (we remark that $p(k)$ is equal to the number of partitions of k into positive integers):

k	1	2	3	4	5	6	7
$p(k)$	1	2	3	5	7	11	15

We list the number of possible Jordan forms, that is the number of similarity classes, for each pair (k_1,k_2):

(k_1,k_2)	$p(k_1)p(k_2)$
$(1,7)$	$1\cdot 15=15$
$(2,6)$	$2\cdot 11=22$
$(3,5)$	$3\cdot 7=21$
$(4,4)$	$5\cdot 5=25$
$(5,3)$	$7\cdot 3=21$
$(6,2)$	$11\cdot 2=22$
$(7,1)$	$15\cdot 1=15$
Total	141

Thus there are 141 similarity classes.

2: Let $A=\begin{pmatrix}2&-7&1&7\\3&-8&2&7\\0&1&-2&-1\\3&-8&3&7\end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is in Jordan form.

Solution: The characteristic polynomial of A is

$$\begin{aligned}\big|A-\lambda I\big| &= \begin{vmatrix}2-\lambda&-7&1&7\\3&-8-\lambda&2&7\\0&1&-2-\lambda&-1\\3&-8&3&7-\lambda\end{vmatrix} = \begin{vmatrix}2-\lambda&-7&1&7\\3&-8-\lambda&2&7\\0&1&-2-\lambda&-1\\0&\lambda&1&-\lambda\end{vmatrix}\\ &= \begin{vmatrix}2-\lambda&-7&1&0\\3&-8-\lambda&2&-1-\lambda\\0&1&-2-\lambda&0\\0&\lambda&1&0\end{vmatrix} = (-1-\lambda)\begin{vmatrix}2-\lambda&-7&1\\0&1&-2-\lambda\\0&\lambda&1\end{vmatrix}\\ &= (-1-\lambda)(2-\lambda)\begin{vmatrix}1&-2-\lambda\\\lambda&1\end{vmatrix} = (\lambda+1)(\lambda-2)(1+2\lambda+\lambda^2)=(\lambda+1)^3(\lambda+2)\end{aligned}$$

so the eigenvalues are $\lambda_1=-1$ and $\lambda_2=2$ with algebraic multiplicities $\dim(K_{-1})=3$ and $\dim(K_2)=1$.

When $\lambda = \lambda_1 = -1$ we have

$$A - \lambda I = A + I = \begin{pmatrix} 3 & -7 & 1 & 7 \\ 3 & -7 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 3 & -8 & 3 & 8 \end{pmatrix} \sim \begin{pmatrix} 3 & -7 & 1 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & -7 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Since $\text{rank}(A+I) = 3$, there is one 3×3 Jordan block for $\lambda_1 = -1$, and since $\dim(K_2) = 1$, there is one 1×1 block for $\lambda_2 = 2$, so the Jordan form of A is

$$P^{-1}AP = J = \begin{pmatrix} -1 & 1 & 0 & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & \\ & & & 2 \end{pmatrix} .$$

Let us find a cycle $\mathcal{C} = \{u_1, u_2, u_3\}$ which is a basis for K_{-1}. We need $u_1 \in \text{Range}(A+I)^2 \cap E_{-1}$. Since $E_{-1} = \text{Null}(A+I)$, which is 1-dimensional, we can choose any $0 \neq u_1 \in E_{-1}$. We choose $u_1 = (0,1,0,1)^T$. We then need $u_2 \in \text{Range}(A - \lambda I)^1$ with $(A+I)u_2 = u_1$ and u_3 with $(A+I)u_3 = u_2$, so we solve $(A+I)x = y$.

$$\big(A+I\big|y\big) = \left(\begin{array}{cccc|c} 3 & -7 & 1 & 7 & y_1 \\ 3 & -7 & 2 & 7 & y_2 \\ 0 & 1 & -1 & -1 & y_3 \\ 3 & -8 & 3 & 8 & y_4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & -7 & 1 & 7 & y_1 \\ 0 & 1 & -1 & -1 & y_3 \\ 0 & 0 & 1 & 0 & -y_1 + y_2 \\ 0 & -1 & 2 & 1 & -y_1 + y_4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & \frac{1}{3}y_1 + \frac{7}{3}y_3 \\ 0 & 1 & -1 & -1 & y_3 \\ 0 & 0 & 1 & 0 & -y_1 + y_2 \\ 0 & 0 & 1 & 0 & -y_1 + y_3 + y_4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{5}{3}y_1 + 2y_2 + \frac{7}{3}y_3 \\ 0 & 1 & 0 & -1 & -y_1 + y_2 + y_3 \\ 0 & 0 & 1 & 0 & -y_1 + y_2 \\ 0 & 0 & 0 & 0 & -y_2 + y_3 + y_4 \end{array}\right) .$$

Thus $(A+I)x = y$ has a solution (so $y \in \text{Range}(A+I)$) if and only if $-y_2 + y_3 + y_4 = 0$, and then the solution is

$$x = \big(-\tfrac{5}{3}y_1 + 2y_2 + \tfrac{7}{3}y_3 \,,\, -y_1 + y_2 + y_3 \,,\, -y_1 + y_2 \,,\, 0 \big)^T + t\,(0,1,0,1)^T .$$

Take $y = u_1$ to get $u_2 = x = (2,1,1,0)^T + t(0,1,0,1)^T = (2, 1+t, 1, t)^T$ for some value of t. Taking $y = u_2 = (2, 1+t, 1, t)^T$ we find that $-y_2 + y_3 + y_4 = 0$ so that $u_2 \in \text{Range}(A - \lambda I)$ for any choice of t. We choose $t = 0$ so that $u_2 = (2,1,1,0)^T$. Taking $y = u_2 = (2,1,1,0)^T$ we get $u_3 = x = (1, 0, -1, 0)^T$.

When $\lambda = \lambda_2 = 2$ we have

$$A - \lambda I = A - 2I = \begin{pmatrix} 0 & -7 & 1 & 7 \\ 3 & -10 & 2 & 7 \\ 0 & 1 & -4 & -1 \\ 3 & -8 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 3 & -8 & 3 & 5 \\ 0 & 1 & -4 & -1 \\ 0 & -7 & 1 & 7 \\ 0 & 2 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & -29 & -3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & -27 & 0 \\ 0 & 0 & 9 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

We need $u_4 \in E_2 = \text{Null}(A - 2I)$, and we can take $u_4 = (1,1,0,1)^T$. Thus we can take

$$P = \big(u_1, u_2, u_3, u_4\big) = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} .$$

3: Let $A = \begin{pmatrix} 1 & 1 & 0 & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \\ & & & -1 & 1 \\ & & & 0 & -1 \end{pmatrix}$.

(a) Find A^n, where n is a positive integer.

Solution: We have $A = \begin{pmatrix} J_3^1 & \\ & J_{-1}^2 \end{pmatrix}$. We can write $J_1^3 = I + N$ where $I = I_{3\times 3}$ and $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Note that $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N^k = 0$ for $k \geq 3$. By the Binomial Theorem (which holds, and can be proven as usual, for $(X+Y)^n$ with $X, Y \in M_{m\times m}$, as long as X and Y commute) we have

$$\left(J_1^3\right)^n = (I+N)^n = I^n + \binom{n}{1} I^{n-1}N + \binom{n}{2} I^{n-2}N^2 + \cdots = I + nN + \tfrac{n(n-1)}{2}N^2 = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we can write $J_{-1}^2 = -I + M$ where $I = I_{2\times 2}$ and $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we note that $M^k = 0$ for $k \geq 2$, then the Binomial Theorem gives

$$\left(J_{-1}^2\right)^n = (-1)^n I + (-1)^{n-1} n\, M = \begin{pmatrix} (-1)^n & (-1)^{n-1}n \\ 0 & (-1)^n \end{pmatrix}.$$

Thus

$$A^n = \begin{pmatrix} \left(J_1^3\right)^n & 0 \\ 0 & \left(J_{-1}^2\right)^n \end{pmatrix} = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} & & \\ 0 & 1 & n & & \\ 0 & 0 & 1 & & \\ & & & (-1)^n & (-1)^{n-1}n \\ & & & 0 & (-1)^n \end{pmatrix}$$

(b) Find $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$.

Solution: Recall that $e^x = \sum\limits_{n=0}^{\infty} \frac{1}{n!}\, x^n$ for all $x \in \mathbb{R}$. Taking the derivative gives $e^x = \sum\limits_{n=0}^{\infty} \frac{n}{n!}\, x^{n-1}$ then taking it again gives $e^x = \sum\limits_{n=0}^{\infty} \frac{n(n-1)}{n!}\, x^{n-2}$. Setting $x = 1$ gives $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$ and $e = \sum\limits_{n=0}^{\infty} \frac{n}{n!}$ and $e = \sum\limits_{n=0}^{\infty} \frac{n(n-1)}{n!}$. Setting $x = -1$ gives $e^{-1} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!}$ and $e^{-1} = \sum\limits_{n=0}^{\infty} \frac{(-1)^{n-1}n}{n!}$. Thus we have

$$e^{J_1^3} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(J_1^3\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum \frac{1}{n!} & \sum \frac{n}{n!} & \sum \frac{n(n-1)}{2n!} \\ 0 & \sum \frac{1}{n!} & \sum \frac{n}{n!} \\ 0 & 0 & \sum \frac{1}{n!} \end{pmatrix} = \begin{pmatrix} e & e & \frac{1}{2}e \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix}$$

and

$$e^{J_{-1}^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(J_{-1}^2\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} (-1)^n & (-1)^{n-1}n \\ 0 & (-1)^n \end{pmatrix} = \begin{pmatrix} \sum \frac{(-1)^n}{n!} & \sum \frac{(-1)^{n-1}n}{n!} \\ 0 & \sum \frac{(-1)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{-1} & e^{-1} \\ 0 & e^{-1} \end{pmatrix}.$$

Thus we have

$$e^A = \begin{pmatrix} e^{J_1^3} & 0 \\ 0 & e^{J_{-1}^2} \end{pmatrix} = \begin{pmatrix} e & e & \frac{1}{2}e & & \\ 0 & e & e & & \\ 0 & 0 & e & & \\ & & & e^{-1} & e^{-1} \\ & & & 0 & e^{-1} \end{pmatrix}.$$

4: Let $A \in M_{n\times n}(\mathbb{C})$.

(a) Show that A is similar to A^T.

Solution: We have $\text{rank}(A - \lambda I)^k = \text{rank}(A^T - \lambda I)^k$ for all $k \in \mathbb{Z}^+$, and so A and A^Thave the same Jordan form.

(b) Show that if A is invertible then there is a matrix $B \in M_{n\times n}(\mathbb{C})$ such that $A = B^2$.

Solution: Let A be invertible. Note that the eigenvalues of A are all nonzero. Choose an invertible matrix P so that $P^{-1}AP = J$ with J in Jordan form. Consider one of the Jordan blocks J_λ^m of J. Write

$$J_\lambda^m = \lambda(I + N)$$

where N is the $m \times m$ matrix whose above-diagonal entries are $N_{i,i+1} = \frac{1}{\lambda}$ and whose other entries are all zero (in the case that $m = 1$, $N = (0)$, the 1×1 zero matrix). Note that N^k has entries $N_{i,i+k} = \frac{1}{\lambda^k}$ and other entries zero, and in particular $N^k = 0$ for all $k \geq m$. By the Binomial Theorem, for $x \in \mathbb{R}$ with $|x| < 1$ we have

$$(1 + x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k.$$

It follows that

$$\left(\sum_{k=0}^{m} \binom{1/2}{k} x^k \right)^2 = 1 + x + \sum_{i=m+1}^{2m} c_i x^i$$

for some $c_i \in \mathbb{R}$ Since $N^k = 0$ for $k > m$ we have

$$\left(\sum_{k=0}^{m} \binom{1/2}{k} N^k \right)^2 = I + N + \sum_{k=m+1}^{2m} c_i N^i = I + N.$$

We choose $\mu \in \mathbb{C}$ with $\mu^2 = \lambda$ and then let

$$S_\mu^m = \mu \sum_{k=0}^{m} \binom{1/2}{k} N^k$$

so that $\left(S_\mu^m\right)^2 = \lambda(I+N) = J_\lambda^m$. We do the same for each Jordan block so that for the block-diagonal matrix $J = \text{diag}\left(J_{\lambda_1}^{m_1}, \cdots, J_{\lambda_l}^{m_l}\right)$ we obtain the block diagonal matrix $S = \text{diag}\left(S_{\mu_1}^{m_1}, \cdots, S_{\mu_l}^{m_l}\right)$ with $S^2 = J$. Finally, we note that $A = PJP^{-1} = PS^2P^{-1} = PSP^{-1}PSP^{-1} = (PSP^{-1})^2$, so we take $B = PSP^{-1}$.

(c) Show that for $n \geq 2$, there is no matrix $B \in M_{n\times n}(\mathbb{C})$ such that $J_0^n = B^2$.

Solution: Suppose, for a contradiction, that $J_0^n = B^2$. Choose an invertible matrix P so that $P^{-1}BP = K$ with K in Jordan form. Note that the eigenvalues of B (hence of K) are all zero (indeed if κ is an eigenvalue of B and w is an eigenvector for κ then $B^2w = B\kappa w = \kappa^2 w$ and so κ^2 is an eigenvalue for $B^2 = J_0^n$). Let b_i be the number of Jordan blocks in K of size at least i. Then

$$\text{rank}(B^2) = \text{rank}(K^2) = \text{rank}(K - 0I)^2 = n - b_1 - b_2.$$

If $b_1 = 1$ then K only has one jordan block, and this block is of size $n \geq 2$ so that $b_2 = 1$, and so we have $\text{rank}(B^2) = n - b_1 - b_2 = n - 2$. If $b_1 \geq 2$ then $\text{rank}(B^2) = n - b_1 - b_2 \leq n - b_1 \leq n - 2$. In either case, we have $\text{rank}(B^2) < n - 1 = \text{rank}(J_0^n)$, giving the desired contradiction.