MATH 245 Linear Algebra 2, Solutions to the Exercises for Chapter 10

1: (a) Find the vertices and the asymptotes of the hyperbola $11x^2 + 24xy + 4y^2 = 500$.

Solution: Let $K(x,y) = 11x^2 + 24xy + 4y^2$. Note that $K(x,y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$ where $A = \begin{pmatrix} 11 & 12 \\ 12 & 4 \end{pmatrix}$. The characteristic polynomial of A is

$$\det(A - xI) = \det \begin{pmatrix} 11-x & 12 \\ 12 & 4-x \end{pmatrix} = x^2 - 15x - 100 = (x-20)(x+5)$$

so the eigenvalues are $\lambda_1 = 20$ and $\lambda_2 = -5$. We have

$$A - \lambda_1 I = \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \sim \begin{pmatrix} -3 & 4 \\ 0 & 0 \end{pmatrix}$$

so we can choose the unit eigenvector $u_1 = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ for λ_1. The other eigenspace will be orthogonal since A is symmetric, so we can choose the unit eigenvector $u_2 = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ for λ_2. Let $P = \begin{pmatrix} u_1, u_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$ and let $D = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$. Then we have $P^*AP = D$. Write $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} s \\ t \end{pmatrix}$ or equivalently $\begin{pmatrix} s \\ t \end{pmatrix} = P^*\begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$K(x,y) = (\,x \quad y\,)\, A \begin{pmatrix} x \\ y \end{pmatrix} = (\,s \quad t\,)\, P^*AP \begin{pmatrix} s \\ t \end{pmatrix} = (\,s \quad t\,)\, D \begin{pmatrix} s \\ t \end{pmatrix} = 20s^2 - 5t^2$$

and so

$$K(x,y) = 500 \iff 20s^2 - 5t^2 = 500 \iff \frac{s^2}{25} - \frac{t^2}{100} = 1\,.$$

This is the hyperbola in the st-plane with vertices at $(\pm 5, 0)$ and asymptotes $s = \pm 2t$. We calculate the points (x,y) corresponding to $(s,t) = (\pm 5, 0)$ (the vertices) and $(s,t) = (1, \pm 2)$ (points on the asymptotes):

$$P\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix} = \pm\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$
$$P\begin{pmatrix} 1 \\ \pm 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ \pm 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 \\ 11 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Thus the curve $K(x,y) = 500$ is the hyperbola with vertices at $(x,y) = \pm(4,3)^T$ and asymptotes $11x + 2y = 0$ and $x + 2y = 0$.

(b) Find the volume of the ellipsoid $2x^2+3y^2+3z^2+2xy+2xz=9$.

Solution: We have

$$2x^2+3y^2+3z^2+2xy+2xz=\big(x\,,\,y\,,\,z\big)A\begin{pmatrix}x\\y\\z\end{pmatrix}\quad\text{where}\quad A=\begin{pmatrix}2&1&1\\1&3&0\\1&0&3\end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned}f_A(x)=\det(A-xI)&=\det\begin{pmatrix}2-x&1&1\\1&3-x&0\\1&0&3-x\end{pmatrix}\\&=-(x-3)^2(x-2)+(x-3)+(x-3)=-(x-3)\big((x-3)(x-2)-2\big)\\&=-(x-s)(x^2-5x+4)=-(x-3)(x-1)(x-4).\end{aligned}$$

The eigenvalues of A are 4, 3 and 1. Since A is symmetric, it is orthogonally diagonalizable, so there is a matrix $P\in M_3(\mathbb{R})$ with $P^TP=I$ such that $P^TAP=D=\mathrm{diag}(4,3,1)$. If we make an orthogonal change of coordinates by letting

$$\begin{pmatrix}r\\s\\t\end{pmatrix}=P\begin{pmatrix}x\\y\\z\end{pmatrix},$$

then in these new coordinates, the equation of the ellipsoid becomes

$$9=(x,y,z)A(x,y,z)^T=(r,s,t)P^TAP(r,s,t)^T=(r,s,t)D(r,s,t)^T=4r^2+3s^2+1t^2\,,$$

that is

$$\frac{r^2}{9/4}+\frac{s^2}{3}+\frac{t^2}{9}=1.$$

This ellipsoid can be obtained from the unit sphere, which has volume $\frac{4}{3}\pi$, by scaling in the orthogonal directions of the r, s and t axes by the factors $\frac{3}{2}$, $\sqrt{3}$ and 3, and so its volume is

$$V=\tfrac{4}{3}\pi\cdot\tfrac{3}{2}\cdot\sqrt{3}\cdot 3=6\sqrt{3}\,\pi.$$

2: (a) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$. Find $\max\limits_{|x|=1} |Ax|$ and $\min\limits_{|x|=1} |Ax|$. Find $u \in \mathbb{R}^3$ with $|u| = 1$ so that $|Au| = \min\limits_{|x|=1} |Ax|$.

Solution: We have

$$A^*A = \begin{pmatrix} 6 & 2 & 5 \\ 2 & 3 & 2 \\ 5 & 2 & 6 \end{pmatrix}.$$

The characteristic polynomial of A^*A is

$$\begin{aligned} f_{A^*A}(x) &= \det\big(A^*A - xI\big) = \det \begin{pmatrix} 6-x & 2 & 5 \\ 2 & 3-x & 2 \\ 5 & 2 & 6-x \end{pmatrix} \\ &= -(x-6)^2(x-3) + 40 + 8(x-6) + 25(x-3) = -(x^2-12x+36)(x-3) + 33x - 83 \\ &= -(x^3 - 15x^2 + 39x - 25) = -(x-1)(x^2 - 14x + 25). \end{aligned}$$

Note that $x^2 - 14x + 25 = 0$ when $x = \frac{14 \pm \sqrt{4\cdot 49 - 4\cdot 25}}{2} = 7 \pm \sqrt{24}$ so the eigenvalues of A^*A, in decreasing order, are $\lambda_1 = 7 + 2\sqrt{6}$, $\lambda_2 = 7 - 2\sqrt{6}$ and $\lambda_3 = 1$, and so the singular values of A are $\sigma_1 = 1 + \sqrt{6}$, $\sigma_2 = -1 + \sqrt{6}$ and $\sigma_3 = 1$. At this stage, we know that

$$\max_{|x|=1} |Ax| = \sigma_1 = 1 + \sqrt{6} \quad \text{and} \quad \min_{|x|=1} |Ax| = \sigma_3 = 1.$$

The minimum value is attained at the unit eigenvector u of A^*A for λ_3. We have

$$A^*A - \lambda_3 I = \begin{pmatrix} 5 & 2 & 5 \\ 2 & 2 & 2 \\ 5 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 5 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so we can take $u = \frac{1}{\sqrt{2}}(-1, 0, 1)^T$.

(b) Define $L : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $L(x_1, x_2, x_3, x_4, \cdots) = \Big(\sum\limits_{i=1}^\infty x_i \,,\, x_2, x_3, x_4, \cdots \Big)$. Find $\sup\limits_{|x|=1} |Lx|$ and $\inf\limits_{|x|=1} |Lx|$.

Solution: Let $a_n = \sum\limits_{i=1}^n \frac{1}{\sqrt{n}} e_i = \big(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots \big)$. Then we have $|a_n| = 1$ and

$$L(a_n) = \sqrt{n}\, e_1 + \sum_{i=2}^n \frac{1}{\sqrt{n}} e_i = \big(\sqrt{n}, \tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}, \cdots, \tfrac{1}{\sqrt{n}}, 0, 0, \cdots \big)$$

so that $\big|L(a_n)\big|^2 = n + \frac{n-1}{n}$. Since $\lim\limits_{n\to\infty} \big|L(a_n)\big| = \infty$ we see that $\sup\limits_{|a|=1} \big|L(a)\big| = \infty$.

Let $b_n = -\sqrt{\frac{n-1}{n}}\, e_1 + \sum\limits_{i=1}^n \frac{1}{n}\, e_{i+1} = \big(-\sqrt{\frac{n-1}{n}}, \frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, 0, 0, \cdots \big)$. Then we have $|b_n| = 1$ and

$$L(b_n) = \Big(1 - \sqrt{\tfrac{n-1}{n}} \Big) e_1 + \sum_{i=1}^n \frac{1}{n}\, e_{i+1} = \Big(1 - \sqrt{\tfrac{n-1}{n}}, \tfrac{1}{n}, \tfrac{1}{n}, \cdots, \tfrac{1}{n}, 0, 0, \cdots \Big)$$

so that $\big|L(b_n)\big|^2 = \Big(1 - \sqrt{\frac{n-1}{n}} \Big)^2 + \frac{1}{n}$. Since $\lim\limits_{n\to\infty} \big|L(b_n)\big| = 0$ we see that $\inf\limits_{|a|=1} \big|L(a)\big| = 0$.

3: Let U and V be non-trivial subspaces of $\mathbb{R}^n$ with $U \cap V = \{0\}$. Recall that

$$\theta(U,V) = \min\left\{\theta(u,v)\,\middle|\,0 \neq u \in U, 0 \neq v \in V\right\}.$$

(a) Show that $\theta(U,V) = \cos^{-1}(\sigma)$ where σ is the largest singular value of the linear map $P : U \to V$ given by $P(x) = \text{Proj}_V(x)$.

Solution: Recall that for fixed $0 \neq u \in \mathbb{R}^n$ we have $\min\limits_{0\neq v\in V} \theta(u,v) = \cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right|$. Thus we have

$$\begin{aligned}\theta(U,V) &= \min_{0\neq u\in U}\ \min_{0\neq v\in V}\ \theta(u,v) = \min_{0\neq u\in U} \cos^{-1}\left(\left|\text{Proj}_V \tfrac{u}{|u|}\right|\right) = \cos^{-1}\left(\max\nolimits_{0\neq u\in U}\left|\text{Proj}_V \tfrac{u}{|u|}\right|\right)\\ &= \cos^{-1}\left(\max_{u\in U, |u|=1}\left|\text{Proj}_V(u)\right|\right) = \cos^{-1}\left(\max\nolimits_{u\in U,|u|=1}\left|P(u)\right|\right) = \cos^{-1}\sigma\end{aligned}$$

where σ is the largest singular value of P.

(b) Let $u_1 = \begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}2\\1\\-1\\2\end{pmatrix}$, $v_1 = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$ and $v_2 = \begin{pmatrix}2\\1\\1\\0\end{pmatrix}$. Let $U = \text{Span}\{u_1,u_2\}$ and $V = \text{Span}\{v_1,v_2\}$. Find $\theta(U,V)$.

Solution: To find the singular values of P we find the matrix of P^*P with respect to orthonormal bases. We apply the Gram-Schmidt Procedure to the basis $\{u_1,u_2\}$ to get

$$w_1 = u_1 = \begin{pmatrix}1\\1\\-1\\1\end{pmatrix},\quad w_2 = u_2 - \frac{u_2 \cdot w_1}{|w_1|^2}\,w_1 = \begin{pmatrix}2\\1\\-1\\2\end{pmatrix} - \frac{6}{4}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}4\\2\\-2\\4\end{pmatrix} - \begin{pmatrix}3\\3\\-3\\3\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}$$

then we normalize to get the orthonormal basis $\mathcal{U} = \{x_1,x_2\}$ with

$$x_1 = \frac{w_1}{|w_1|} = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix},\quad x_2 = \frac{w_2}{|w_2|} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}.$$

Next we apply the Gram-Schmidt Procedure to $\{v_1,v_2\}$ to get

$$z_1 = v_1 = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix},\quad z_2 = v_2 - \frac{v_2 \cdot z_1}{|z_1|^2}\,z_1 = \begin{pmatrix}2\\1\\1\\0\end{pmatrix} - \frac{3}{3}\begin{pmatrix}1\\1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}$$

then we normalize to get the orthonormal basis $\mathcal{V} = \{y_1,y_2\}$ for V where

$$y_1 = \frac{z_1}{|z_1|^2} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\0\\-1\end{pmatrix},\quad y_2 = \frac{z_2}{|z_2|^2} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix}.$$

Now let

$$A = [P]_{\mathcal{V}}^{\mathcal{U}} = \begin{pmatrix}x_1\cdot y_1 & x_2\cdot y_1\\ x_1\cdot y_2 & x_2\cdot y_2\end{pmatrix} = \tfrac{1}{2\sqrt{3}}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}.$$

Then

$$[P^*P]_{\mathcal{V}}^{\mathcal{U}} = A^*A = \frac{1}{12}\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix} = \tfrac{1}{12}\begin{pmatrix}2 & 2\\ 2 & 10\end{pmatrix} = \tfrac{1}{6}\,B\ ,\ \text{where } B = \begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}.$$

The characteristic polynomial of B is

$$\det(B - tI) = \det\begin{pmatrix}1-t & 1\\ 1 & 5-t\end{pmatrix} = t^2 - 6t + 4$$

so B has eigenvalues $\lambda = \frac{6\pm\sqrt{20}}{2} = 3 \pm \sqrt{5}$. Since $A^*A = \frac{1}{6}\,B$, the eigenvalues of A^*A, or equivalently the eigenvalues of P^*P, are $\frac{3\pm\sqrt{5}}{6}$. Thus the largest singular value of P is $\sigma = \sqrt{\frac{3+\sqrt{5}}{6}}$ and we obtain

$$\theta(U,V) = \cos^{-1}\sqrt{\tfrac{3+\sqrt{5}}{6}}\,.$$