MATH 245 Linear Algebra 2, Exercises for Chapter 1

1: Let $P = a + \text{Null}(A)$ and $Q = b + \text{Col(B)}$ where

$$a = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix} \ , \ b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \ , \ A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 3 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 3 & -1 \end{pmatrix} .$$

Find a point $p \in \mathbb{R}^4$ and a basis for a subspace $U \subseteq \mathbb{R}^4$ such that $P \cap Q = p + U$.

2: (a) Show that the set $A = \left\{(x,y) \in \mathbb{R}^2 \middle| x^2 + y^2 \geq (x^2 + y^2 - x)^2\right\}$ is not convex. (Hint: consider polar coordinates).

(b) Show that the set $B = \left\{(x,y) \in \mathbb{R}^2 \middle| y \geq x^2\right\}$ is convex.

3: Let W be a vector space over $\mathbb{R}$. A nonempty set $\emptyset \neq C \subseteq W$ is called **conical** when it has the property that

$$\text{for all } a \in C \text{ and all } 0 \leq t \in \mathbb{R}, \text{ we have } ta \in C.$$

(a) Show that the intersection of a set of conical sets in W is conical.

(b) For a nonempty set $\emptyset \neq \mathcal{S} \subseteq W$ we define the **convex cone** of $\mathcal{S}$, denoted by $\text{Cone}(\mathcal{S})$, to be the smallest convex conical subset of W which contains $\mathcal{S}$, or equivalently the intersection of all convex conical sets in W which contain $\mathcal{S}$. Show that

$$\text{Cone}(\mathcal{S}) = \left\{ \sum_{i=0}^{n} t_i a_i \middle| n \in \mathbb{N}, a_i \in \mathcal{S}, 0 \leq t_i \in \mathbb{R} \right\}.$$

4: Let V and W be vector spaces over a field F in which $2 \neq 0$. Here is an alternate definition for an affine space in V. Let us say that a nonempty set $\emptyset \neq P \subseteq V$ is **affine** in V when $sx + ty \in P$ for all $x, y \in P$ and for all $s, t \in F$ with $s + t = 1$. Also, let us say that a map $A : V \to W$ is **affine** when

$$A(sx + ty) = s\,A(x) + t\,A(y) \text{ for all } x, y \in V \text{ and } s, t \in F \text{ with } s + t = 1.$$

(a) For $\emptyset \neq P \subseteq V$, show that P is affine if and only if $P = a + U$ for some $a \in V$ and some subspace $U \subseteq V$.

(b) Show that the affine maps $A : V \to W$ are the maps of the form $A(x) = a + L(x)$ for some point $a \in W$ and some linear map $L : V \to W$.

(c) Show that when $F = \mathbb{R}$, if $A : V \to W$ is affine and $C \subseteq V$ is convex, then the image $A(C)$ is convex.

5: (a) Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{R}^n$ and let $x \in [\mathcal{S}]$. Show that $x = \sum_{i=0}^{m} t_i a_i$ for some $m \in \mathbb{N}$ with $m \leq n$, some $a_i \in \mathcal{S}$, and some $0 \leq t_i \in \mathbb{R}$ with $\sum t_i = 1$.

(b) Let $\mathcal{S} \subseteq \mathbb{R}^n$ with $|\mathcal{S}| \geq n + 2$. Show that there exist disjoint, nonempty subsets $A, B \subseteq \mathcal{S}$ such that $[A] \cap [B] \neq \emptyset$.

6: For $x, y \in \mathbb{R}^n$, write $x \leq y$ when $x_i \leq y_i$ for all i. Let $P = \left\{x \in \mathbb{R}^4 \middle| Ax = a \text{ and } Bx \leq b\right\}$ where

$$a = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \ , \ b = \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} \ , \ A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 3 \\ 1 & 2 & 0 & -1 \\ 2 & 1 & -1 & -2 \end{pmatrix} .$$

Find a set of distinct points $a_0, a_1, a_2, a_3 \in \mathbb{R}^4$ such that $P = [a_0, a_1, a_2, a_3]$.