6. Orthogonal Bases, Orthogonal Complement and Orthogonal Projection

6.1 Definition: Let W be an inner product space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. For a subset $\mathcal{A}\subseteq W$, we say that $\mathcal{A}$ is **orthogonal** when $\langle u,v\rangle=0$ for all $u,v\in\mathcal{A}$ with $u\neq v$, and we say that $\mathcal{A}$ is **orthonormal** when $\mathcal{A}$ is orthogonal with $\|u\|=1$ for every $u\in\mathcal{A}$.

6.2 Example: Let $u_1,u_2,\cdots,u_l\in\mathbb{C}^n$ and let $A=\big(u_1,\cdots,u_l\big)\in M_{n\times l}(\mathbb{C})$. Since

$$A^*A=\begin{pmatrix}{u_1}^*\\ \vdots\\ {u_l}^*\end{pmatrix}\begin{pmatrix}u_1&\cdots&u_l\end{pmatrix}=\begin{pmatrix}\langle u_1,u_1\rangle&\langle u_2,u_1\rangle&\cdots&\langle u_l,u_l\rangle\\ \vdots&\vdots&&\vdots\\ \langle u_1,u_l\rangle&\langle u_2,u_l\rangle&\cdots&\langle u_l,u_l\rangle\end{pmatrix}$$

it follows that $\{u_1,\cdots,u_l\}$ is orthogonal if and only if A^*A is diagonal, and $\{u_1,\cdots,u_l\}$ is orthonormal if and only if $A^*A=I$.

6.3 Example: Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ and let $a_0,a_1,\cdots,a_n$ be distinct points in $\mathbb{F}$. Consider the vector space $P_n=P_n(\mathbb{F})$ with the inner product $\langle f,g\rangle=\sum\limits_{i=0}^n f(a_i)\overline{g(a_i)}$. For each index k, let $g_k\in P_n$ be given by

$$g_k(x)=\frac{\prod_{i\neq k}(x-a_i)}{\prod_{i\neq k}(a_k-a_i)}$$

so that $g_k(a_i)=\delta_{k,i}$. For $f\in P_n$ we have $\langle f,g_k\rangle=\sum\limits_{i=0}^n f(a_i)\overline{g(a_i)}=\sum\limits_{i=0}^n f(a_i)\delta_{k,i}=f(a_k)$. In particular, we have $\langle g_j,g_k\rangle=g_j(a_k)=\delta_{j,k}$ and so the set $\{g_0,g_1,\cdots,g_n\}$ is an orthonormal basis for $P_n(\mathbb{F})$.

6.4 Theorem: *Let W be an inner product space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}\subseteq W$.*

(1) If $\mathcal{A}$ is an orthogonal set of nonzero vectors then for $x\in\operatorname{Span}\mathcal{A}$ with say $x=\sum\limits_{i=1}^n t_iu_i$ where $t_i\in\mathbb{F}$ and $u_i\in\mathcal{A}$, we have

$$t_k=\frac{\langle x,u_k\rangle}{\|u_k\|^2}$$

for all indices k, and in particular, $\mathcal{A}$ is linearly independent.

(2) If $\mathcal{A}$ is orthonormal then for $x\in\operatorname{Span}\mathcal{A}$ with say $x=\sum\limits_{i=1}^n t_iu_i$ where $t_i\in\mathbb{F}$ and $u_i\in\mathcal{A}$, we have $t_k=\langle x,u_k\rangle$ for all k, and in particular, $\mathcal{A}$ is linearly independent.

Proof: To prove Part (1), suppose that $\mathcal{A}$ is an orthogonal set of nonzero vectors and let $x=\sum\limits_{i=1}^n t_iu_i$ with each $t_i\in\mathbb{F}$ and each $u_i\in\mathcal{A}$. Then for all indices k, since $\langle u_i,u_k\rangle=0$ whenever $i\neq k$ we have $\langle x,u_k\rangle=\Big\langle\sum\limits_{i=1}^n t_iu_1\,,\,u_k\Big\rangle=\sum\limits_{i=1}^n t_i\langle u_i,u_k\rangle=t_k\langle u_k,u_k\rangle=t_k\|u_k\|^2$ and so $t_k=\frac{\langle x,u_k\rangle}{\|u_k\|^2}$, as required. In particular, when $x=0$ we find that $t_k=0$ for all k, and this shows that $\mathcal{A}$ is linearly independent. This proves Part (1), and Part (2) follows immediately from Part (1).

6.5 Theorem: *(The Gram-Schmidt Procedure) Let W be a finite or countable dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A} = \{u_1, u_2, \cdots\}$ be an ordered basis for W. Let $v_1 = u_1$ and for $k \geq 2$ let*

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i \rangle}{\|v_i\|^2} v_i.$$

Then the set $\mathcal{B} = \{v_1, v_2, \cdots\}$ is an orthogonal basis for W with the property that for every index $k \geq 1$ we have $\text{Span}\{v_1, \cdots, v_k\} = \text{Span}\{u_1, \cdots, u_k\}$.

Proof: We prove, by induction on k, that $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal basis for $\text{Span}\{u_1, u_2, \cdots, u_k\}$. When $k = 1$ this is clear since $v_1 = u_1$. Let $k \geq 2$ and suppose, inductively, that $\{v_1, \cdots, v_{k-1}\}$ is an orthogonal basis for $\text{Span}\{u_1, \cdots, u_{k-1}\}$. Since $v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i \rangle}{\|v_i\|^2} v_i$, we see that u_k is equal to v_k plus a linear combination of the vectors $v_1, \cdots, v_{i-1}$, and so we have $\text{Span}\{v_1, \cdots, v_{k-1}, v_k\} = \text{Span}\{v_1, \cdots, v_{k-1}, u_k\}$. By the induction hypothesis, we have $\text{Span}\{v_1, \cdots, v_{k-1}\} = \text{Span}\{u_1, \cdots, u_{k-1}\}$ so we have

$$\text{Span}\{v_1, \cdots, v_{k-1}, v_k\} = \text{Span}\{v_1, \cdots, v_{k-1}, u_k\} = \text{Span}\{u_1, \cdots, u_{k-1}, u_k\}.$$

It remains to show that the set $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal set. By the induction hypothesis, we have $\langle v_j, v_i \rangle = 0$ for all $1 \leq i < j < k$, so it suffices to show that $\langle v_k, v_j \rangle = 0$ for all indices $1 \leq j < k$ and indeed, for $1 \leq j < k$ we have

$$\begin{aligned} \langle v_k, v_j \rangle &= \left\langle u_k - \sum_{i=1}^{k-1} \tfrac{\langle u_k, v_i \rangle}{\|v_i\|^2} v_i \ , \ v_j \right\rangle = \langle u_k, v_j \rangle - \sum_{i=1}^{k-1} \tfrac{\langle u_k, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle \\ &= \langle u_k, v_j \rangle - \frac{\langle u_k, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle = 0. \end{aligned}$$

6.6 Corollary: *Every finite or countable dimensional inner product space W over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ has an orthonormal basis.*

Proof: Let W be a finite or countable dimensional inner product space over $f = \mathbb{R}$ or $\mathbb{C}$. Choose an ordered basis $\mathcal{A} = \{u_1, u_2, \cdots\}$ for W. Apply the Gram-Schmidt Procedure to the basis $\mathcal{A}$ to obtain an orthogonal basis $\mathcal{B} = \{v_1, v_2, \cdots\}$ for W. For each index $k \geq 1$, let $w_k = \frac{v_k}{\|v_k\|}$. Then $\mathcal{C} = \{w_1, w_2, \cdots\}$ is an orthonormal basis for W.

6.7 Remark: It is *not* the case that every uncountable dimensional inner product space has an orthonormal basis.

6.8 Corollary: *Let W be a finite or countable dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subseteq W$ be a finite dimensional subspace. Then every orthogonal (or orthonormal) basis for U extends to an orthogonal (or orthonormal) basis for W.*

Proof: Let $\mathcal{A} = \{u_1, u_2, \cdots, u_l\}$ be an ordered orthogonal (or orthonormal) basis for U. Extend $\mathcal{A}$ to an ordered basis $\{u_1, \cdots, u_l, v_1, v_2, \cdots\}$ for W. Apply the Gram-Schmidt Procedure to this basis to obtain an orthogonal basis $\mathcal{C} = \{u_1{}', \cdots, u_l{}', w_1, w_2\}$ for W. Verify that since $\{u_1, \cdots, u_l\}$ is already orthogonal, it follows that the vectors u_i are left unchanged in the Gram Schmidt Procedure so that in fact $u_i{}' = u_i$ for all indices i, and so the new orthogonal basis $\mathcal{C}$ extends the original orthogonal basis $\mathcal{A}$.

6.9 Remark: The above corollary does *not* hold in general in the case that the subspace U is countable dimensional, as we shall soon see in Example 6.16.

6.10 Corollary: *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let U and V be finite or countable dimensional inner product spaces over $\mathbb{F}$. Then U and V are isomorphic (as inner product spaces) if and only if $\dim(U) = \dim(V)$. In particular, if $\dim(U) = n$ then U is isomorphic to $\mathbb{F}^n$ and if $\dim(U) = \aleph_0$ then U is isomorphic to $\mathbb{F}^\infty$.*

Proof: Suppose that U and V are isomorphic. Let $L : U \to V$ be an isomorphism. Let $\mathcal{A} = \{u_1, u_2, \cdots\}$ be any basis for U. Since L is a bijective linear map, it follows that $\mathcal{B} = \{L(u_1), L(u_2), \cdots\}$ is a basis for V, and that $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality. Thus $\dim(U) = \dim(V)$.

Conversely, suppose $\dim(U) = \dim(V)$. Let $\mathcal{A} = \{u_1, u_2, \cdots\}$ and $\mathcal{B} = \{v_1, v_2, \cdots\}$ be orthonormal bases for U and V. Let $L : U \to V$ be the (unique) bijective linear map with $L(u_i) = v_i$ for all i. Then L preserves inner product because for $x, y \in U$ with say $x = \sum\limits_{i\geq 1} t_i u_i$ and $y = \sum\limits_{j\geq 1} t_j u_j$ we have

$$\langle x, y\rangle = \Big\langle \sum_{i\geq 1} s_i u_i \,,\, \sum_{j\geq 1} t_j u_j \Big\rangle = \sum_{i\geq 1, j\geq 1} s_i \overline{t_j} \langle u_i, u_j\rangle = \sum_{i\geq 1, j\geq 1} s_i \overline{t_j}\, \delta_{i,j} = \sum_{i\geq 1} s_i \overline{t_i}$$

and

$$\begin{aligned}\big\langle L(x), L(y)\big\rangle &= \Big\langle L\big(\sum_{i\geq 1} s_i u_i\big) \,,\, L\big(\sum_{j\geq 1} t_j u_j\big)\Big\rangle = \Big\langle \sum_{i\geq 1} s_i L(u_i) \,,\, \sum_{j\geq 1} t_j L(u_j)\Big\rangle \\ &= \Big\langle \sum_{i\geq 1} s_i v_i \,,\, \sum_{j\geq 1} t_j v_j \Big\rangle = \sum_{i\geq 1, j\geq 1} s_i \overline{t_j} \langle v_i, v_j\rangle = \sum_{i\geq 1, j\geq 1} s_i \overline{t_j}\, \delta_{i,j} = \sum_{i\geq 1} s_i \overline{t_j}\,.\end{aligned}$$

6.11 Corollary: *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let U be an n-dimensional inner product space over $\mathbb{F}$, and let $\mathcal{A} = \{u_1, \cdots, u_n\}$ be an orthonormal basis for U. Then the map $\phi_\mathcal{A} : U \to \mathbb{F}^n$ given by $\phi_\mathcal{A}(x) = [x]_\mathcal{A}$ is an isomorphism. In particular, when $x = \sum\limits_{i=1}^n s_i u_i$ and $y = \sum\limits_{i=1}^n t_i u_i$ so that $s = [x]_\mathcal{A}$ and $t = [y]_\mathcal{A}$, we have $\langle x, y\rangle = \langle s, t\rangle = t^* s$.*

Proof: Taking $V = \mathbb{F}^n$ with its standard orthonormal basis $\mathcal{B} = \{e_1, \cdots, e_n\}$, the map $L : U \to V$ with $L(u_i) = e_i$, used in the above proof, is precisely the map $\phi_\mathcal{A}$.

6.12 Definition: Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For a subspace $U \subseteq W$, we define the **orthogonal complement** of U in W to be the set

$$U^\perp = \big\{ x \in W \big| \langle x, u \rangle = 0 \text{ for all } u \in U \big\}.$$

6.13 Theorem: *Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subseteq W$ be a subspace. Then*

(1) $U^\perp$ is a subspace of W,
(2) if $\mathcal{A}$ is a basis for U then $U^\perp = \big\{ x \in W \big| \langle x, u \rangle = 0 \text{ for all } u \in \mathcal{A} \big\}$,
(3) $U \cap U^\perp = \{0\}$, and
(4) $U \subseteq (U^\perp)^\perp$.
(5) If U is finite dimensional then $U \oplus U^\perp = W$, and
(6) If $U \oplus U^\perp = W$ then $U = (U^\perp)^\perp$.

Proof: We leave the proofs of Parts (1) to (4) as an exercise (they are identical to the proofs of analogous parts of Theorem 2.11). To prove Parts (5), suppose that U is finite dimensional. Let $\mathcal{A} = \{u_1, u_2, \cdots, u_l\}$ be an orthonormal basis for $\mathcal{A}$. To prove Part (5), we need to show that for every $x \in W$ there exist unique vectors $u, v \in W$ with $u \in U$, $v \in V$ and $u + v = x$. First we prove uniqueness. Let $x \in W$, and suppose that $u \in U$, $v \in U^\perp$ and $u + v = x$. Note that for all indices i we have

$$\langle x, u_i \rangle = \langle u + v, u_i \rangle = \langle u, u_i \rangle + \langle v, u_i \rangle = \langle u, u_i \rangle.$$

and so, by Theorem 6.4, we have

$$u = \sum_{i=1}^{l} \langle u, u_i \rangle u_i = \sum_{i=1}^{l} \langle x, u_i \rangle u_i .$$

This proves uniqueness, since given $x \in W$, the vector u must be given by $u = \sum\limits_{i=1}^{l} \langle x, u_i \rangle u_i$ and then the vector v must be given by $v = x - u$.

To prove existence, let $x \in W$ and choose u and v to be the vectors $u = \sum\limits_{i=1}^{l} \langle x, u_i \rangle u_i$ and $v = x - u$. Then we have $u \in U$ and $u + v = x$, so it suffices to show that $v \in U^\perp$. For all indices k we have

$$\begin{aligned} \langle v, u_k \rangle = \langle x - u, u_k \rangle = \langle x, u_k \rangle - \langle u, u_k \rangle &= \langle x, u_k \rangle - \Big\langle \sum_{i=1}^{l} \langle x, u_i \rangle u_i \,,\, u_k \Big\rangle \\ = \langle x, u_k \rangle - \sum_{i=1}^{l} \langle x, u_i \rangle \langle u_i, u_k \rangle &= \langle x, u_k \rangle - \sum_{i=1}^{l} \langle x, u_i \rangle \delta_{i,k} = \langle x, u_k \rangle - \langle x, u_k \rangle = 0. \end{aligned}$$

Since $\langle v, u_k \rangle = 0$ for all $1 \le k \le l$, from Part (2) we have $v \in U^\perp$. This proves Part (5).

To prove Part (6), suppose that $U \oplus U^\perp = W$. From Part (4), we have $U \subseteq (U^\perp)^\perp$. Conversely, let $x \in (U^\perp)^\perp$. Since $U \oplus U^\perp = W$ we can choose $u, v \in W$ with $u \in U$, $v \in U^\perp$ and $u + v = x$. Since $x \in (U^\perp)^\perp$ and $u \in U \subseteq (U^\perp)^\perp$, we have $v = x - u \in (U^\perp)^\perp$. Thus $v \in U^\perp \cap (U^\perp)^\perp$. By Part 3, $U^\perp \cap (U^\perp)^\perp = \{0\}$ and so $v = 0$. Thus $x = u + v = u \in U$.

6.14 Example: As an exercise, show that for $A \in M_{n\times l}(\mathbb{C})$, using the standard inner product in $\mathbb{C}^n$ we have $(\text{Null}A)^\perp = \text{Col}A^*$ and $(\text{Col}A)^\perp = \text{Null}A^*$.

6.15 Remark: When U is an infinite dimensional subspace of W, we do not always have $U \oplus U^\perp = W$ and we do not always have $(U^\perp)^\perp = U$, as the following example shows.

6.16 Example: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $W = \mathbb{F}^\infty$. Let $U = \left\{a = (a_0, a_1, \cdots)\middle| \sum_{i=0}^{\infty} a_i = 0\right\}$. Note that U is a proper subspace of W and it is countable dimensional with countable basis $\mathcal{A} = \{u_1, u_2, \cdots\}$ where $u_k = e_k - e_0 = (-1, 0, \cdots, 0, 1, 0, 0, \cdots)$. Although $U \subsetneq W$ we have

$$\begin{aligned}U^\perp &= \left\{x \in W \middle| \langle x, u_k\rangle = 0 \text{ for all } k\right\} = \left\{x \in W \middle| \langle x, e_k - e_0\rangle = 0 \text{ for all } k\right\} \\ &= \left\{x \in W \middle| x_k = x_0 \text{ for all } k\right\} = \left\{(x_0, x_1, \cdots) \in W \middle| x_0 = x_1 = x_2 = \cdots\right\} = \{0\}\end{aligned}$$

because for $(x_0, x_1, \cdots) \in W$ we have $x_0 = 0$ for all but finitely many indices i. Notice that in this example we do not have $U \oplus U^\perp = W$. Also notice that, although we could apply the Gram-Schmidt Procedure to the basis $\mathcal{A}$ to obtain an orthogonal basis $\mathcal{B} = \{v_0, v_1, \cdots\}$ for U, the basis $\mathcal{B}$ cannot be extended to an orthogonal basis for W because there is no nonzero vector $0 \neq x \in W$ with $\langle x, v_i\rangle = 0$ for all i.

6.17 Definition: Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subseteq W$ be a finite dimensional subspace (or, more generally, let $U \subseteq W$ be any subspace such that $U \oplus U^\perp = W$). For $x \in W$, we define the **orthogonal projection** of x onto U, denoted by $\text{Proj}_U(x)$, as follows. Since $W = U \oplus U^\perp$, we can choose unique vectors $u, v \in W$ with $u \in U$, $v \in U^\perp$ and $u + v = x$. We then define

$$\text{Proj}_U(x) = u.$$

Since $U = (U^\perp)^\perp$, for u and v as above we have $\text{Proj}_{U^\perp}(x) = v$. When $y \in W$ and $U = \text{Span}\,\{y\}$, we also write $\text{Proj}_y(x) = \text{Proj}_U(x)$.

6.18 Note: Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be a finite dimensional subspace of W. Let $\mathcal{A} = \{u_1, u_2, \cdots, u_l\}$ be an orthogonal basis for U. Then for $x \in W$, as in the proof of Part (5) of Theorem 6.13, we see that

$$\text{Proj}_U(x) = \sum_{i=1}^{l} \frac{\langle x, u_i\rangle}{\|u_i\|^2}\, u_i.$$

6.19 Example: As an exercise, show that for $A \in M_{n\times l}(\mathbb{C})$ and $U = \text{Col}A$, given $x \in \mathbb{C}^n$ there exists $t \in \mathbb{C}^l$ such that $A^*At = A^*x$ and that for any such t we have $\text{Proj}_U(x) = At$. In particular, when $\text{rank}(A) = l$ show that A^*A is invertible so that $\text{Proj}_U(x) = A(A^*A)^{-1}A^*x$.

6.20 Theorem: *Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subseteq W$ be a finite dimensional subspace (or, more generally, a subspace such that $U \oplus U^\perp = W$). Let $x \in W$. Then $\mathrm{Proj}_U(x)$ is the unique point in U which is nearest to x.*

Proof: Let $u, v \in W$ be the vectors with $u \in U$, $v \in U^\perp$ and $u + v = x$, so that we have $\mathrm{Proj}_U(x) = u$. Let $w \in U$ with $w \neq u$. Since $\langle w-u, x-u\rangle = \langle w-u, v\rangle = \langle w, v\rangle - \langle u, v\rangle = 0$, Pythagoras' Theorem gives

$$\|x - w\|^2 = \|(x-u) - (w-u)\|^2 = \|x-u\|^2 + \|w-u\|^2 > \|x-u\|^2$$

and so $\|x - w\| > \|x - u\|$.

6.21 Example: Find the quadratic polynomial $f \in P_2 = P_2(\mathbb{R})$ which minimizes

$$\int_{-1}^{1} \big(f(x) - \|x\|\big)^2\, dx.$$

Solution: Let $W = \mathcal{C}^0\big([-1,1], \mathbb{R}\big)$ with inner product given by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$. Then we need to find the polynomial $f \in P_2$ which minimizes $\mathrm{dist}(f, g)$ where $g(t) = |t|$, so we must take

$$f = \mathrm{Proj}_{P_2}(g).$$

Let $p_0 = 1$, $p_1 = x$ and $p_2 = x^2$ so that $\{p_0, p_1, p_2\}$ is the standard basis for P_2. Apply the Gram-Schmidt Procedure to get

$$\begin{aligned}
q_0 &= p_0 = 1, \\
q_1 &= p_1 - \frac{\langle p_1, q_0\rangle}{\|q_0\|^2}\, q_0 = x - \frac{\langle x, 1\rangle}{\|1\|^2} \cdot 1 = x - \frac{\int_{-1}^{1} x\, dx}{\int_{-1}^{1} 1\, dx} \cdot 1 = x - \tfrac{0}{2} \cdot 1 = x, \\
q_2 &= p_2 - \frac{\langle p_2, q_0\rangle}{\|q_0\|^2}\, q_0 - \frac{\langle p_2, q_1\rangle}{\|q_1\|^2}\, q_1 = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2} \cdot 1 - \frac{\langle x^2, x\rangle}{\|x\|^2} \cdot x \\
&= x^2 - \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} 1\, dx} \cdot 1 - \frac{\int_{-1}^{1} x^3\, dx}{\int_{-1}^{1} x^2\, dx} \cdot x = x^2 - \tfrac{2/3}{2} \cdot 1 - \tfrac{0}{2/3} \cdot x = x^2 - \tfrac{1}{3}.
\end{aligned}$$

Using the orthogonal basis $\{q_0, q_1, q_2\} = \left\{1, x, x - \frac{1}{3}\right\}$ for P_2, we calculate

$$\begin{aligned}
f = \mathrm{Proj}_{P_2}(g) &= \frac{\langle g, q_0\rangle}{\|q_0\|}\, q_0 + \frac{\langle g, q_1\rangle}{\|q_1\|^2}\, q_1 + \frac{\langle g, q_2\rangle}{\|q_2\|^2}\, q_2 \\
&= \frac{\langle \|x\|, 1\rangle}{\|1\|^2} \cdot 1 + \frac{\langle \|x\|, x\rangle}{\|x\|^2} \cdot x + \frac{\langle \|x\|, x^2 - \frac{1}{3}\rangle}{\|x^2 - \frac{1}{3}\|^2} \cdot (x - \tfrac{1}{3}) \\
&= \frac{\int_{-1}^{1} \|x\|\, dx}{\int_{-1}^{1} 1\, dx} \cdot 1 + \frac{\int_{-1}^{1} x\|x\|\, dx}{\int_{-1}^{1} x^2\, dx} \cdot x + \frac{\int_{-1}^{1} \|x\|(x^2 - \frac{1}{3})\, dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\, dx} \cdot (x - \tfrac{1}{3}) \\
&= \tfrac{1}{2} \cdot 1 + \tfrac{0}{2/3} \cdot x + \frac{\frac{1}{2} - \frac{1}{3}}{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} \cdot (x^2 - \tfrac{1}{3}) = \tfrac{1}{2} + \tfrac{1/6}{8/45}(x^2 - \tfrac{1}{3}) \\
&= \tfrac{1}{2} + \tfrac{15}{16}(x^2 - \tfrac{1}{3}) = \tfrac{3}{16} + \tfrac{15}{16} x^2.
\end{aligned}$$