11. Tensor Algebras

11.1 Definition: Recall that for a vector space U over a field $\mathbb{F}$, we define the **dual space** of U to be the vector space

$$U^* = \big\{\text{linear maps } L : U \to \mathbb{F}\big\}.$$

Recall also, that when U is finite dimensional and $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$ is a basis for U, we can define linear maps $f_i : U \to \mathbb{F}$ for $i = 1, 2, \cdots, n$ by requiring that $f_i(u_j) = \delta_{ij}$, and then $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ is a basis for U^* which is called the **dual basis** to $\mathcal{U}$. We shall sometimes identify the double dual U^{**} with U by identifying the element $u \in U$ with the linear map $u : U^* \to \mathbb{F}$ given by $u(f) = f(u)$.

11.2 Definition: Let $U_1, U_2, \cdots, U_k$ and V be vector spaces over a field $\mathbb{F}$. A map

$$L : U_1 \times U_2 \times \cdots \times U_k \to V$$

is called k-linear when

$$L(u_1, \cdots, t\, u_i, \cdots, u_k) = t\, L(u_1, \cdots, u_i, \cdots, u_k) \text{ , and}$$
$$L(u_1, \cdots, v + w, \cdots, u_k) = L(u_1, \cdots, v, \cdots, u_k) + L(u_1, \cdots, w, \cdots, u_k)$$

for all indices i, all vectors $u_1, \cdots, u_k, v, w$ in the appropriate vector spaces, and all $t \in \mathbb{F}$. When $U_1, U_2, \cdots, U_k$ are finite dimensional, the **tensor product** of $U_1, U_2, \cdots, U_k$ is the vector space

$$U_1 \otimes U_2 \otimes \cdots \otimes U_k = \Big\{ k\text{-linear maps } L : U_1^* \times U_2^* \times \cdots \times U_k^* \to \mathbb{F}\Big\}.$$

For $u_1, u_2, \cdots, u_k$ with each $u_i \in U_i$, we define $(u_1 \otimes u_2 \otimes \cdots \otimes u_k) \in U_1 \otimes U_2 \otimes \cdots \otimes U_k$ by

$$(u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) = g_1(u_1) g_2(u_2) \cdots g_k(u_k),$$

where each $g_i \in U_i^*$.

11.3 Example: The dot product $\,\bullet\, : (\mathbb{F}^n)^2 \to \mathbb{F}$ given by $u \bullet v = v^T u$ is a 2-linear map.

11.4 Example: An inner product $\langle\ ,\ \rangle : U^2 \to \mathbb{R}$ on a vector space U over $\mathbb{R}$ is 2-linear.

11.5 Example: The determinant $D : (\mathbb{F}^n)^n \to \mathbb{F}$ given by $D(u_1, u_2, \cdots, u_n) = \det(A)$, where $A = (u_1, u_2, \cdots, u_n) \in M_{n\times n}(\mathbb{F})$, is an n-linear map.

11.6 Example: The generalized cross product $X : (\mathbb{F}^n)^{n-1} \to \mathbb{F}$ is $(n-1)$-linear.

11.7 Theorem: *Let $U_1, U_2, \cdots, U_k$ be finite dimensional vector spaces. For each index i, let $\mathcal{U}_i$ be a basis for U_i. Then the set*

$$\left\{u_1 \otimes u_2 \otimes \cdots \otimes u_k \,\middle|\, \text{each } u_i \in \mathcal{U}_i\right\}$$

is a basis for $U_1 \otimes U_2 \otimes \cdots \otimes U_k$. In particular $\dim(U_1 \otimes U_2 \otimes \cdots \otimes U_k) = \prod_{i=1}^{k} \dim(U_i)$.

Proof: Let $\mathcal{U}_i = \{u_{i1}, u_{i2}, \cdots, u_{i,n_i}\}$ be a basis for U_i and let $\mathcal{F}_i = \{f_{i1}, f_{i2}, \cdots, f_{i,n_i}\}$ be the dual basis for U_i^*. Then for appropriate indices $i_1, i_2, \cdots, i_k$ and $j_1, j_2, \cdots, j_k$ (that is for $1 \leq i_1 \leq n_1$, $1 \leq i_2 \leq n_2$, $\cdots$, $1 \leq i_k \leq n_k$ and similarly for $j_1, j_2, \cdots, j_k$) we have

$$\begin{aligned}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(f_{1,j_1}, f_{2,j_2}, \cdots, f_{k,j_k}) &= f_{1,j_1}(u_{1,i_1}) f_{2,j_2}(u_{2,i_2}) \cdots f_{k,i_k}(u_k, i_k) \\ &= \delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_k,j_k} = \begin{cases} 1 \text{ if } i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k \\ 0 \text{ otherwise.} \end{cases}\end{aligned}$$

It follows that the set of elements of the form $(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ is linearly independent because if $0 = \alpha = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots, \otimes u_{k,i_k})$ then for all appropriate choices of indices $j_1, j_2, \cdots, j_k$ we have

$$0 = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes \cdots \otimes u_{k,i_k})(f_{1,j_1}, \cdots, f_{k,j_k}) = a_{j_1 j_2 \cdots j_k}$$

More generally, for $g_i \in U_i^*$ with say $g_i = \sum c_{ij} f_{ij}$ we have

$$\begin{aligned}&(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(g_1, g_2, \cdots, g_k) \\ &\quad = (u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})\Big(\sum_{j_1} c_{1,j_1} f_{1,j_1}, \sum_{j_2} c_{2,j_2} f_{2,j_2}, \cdots, \sum_{j_k} c_{2,j_2} f_{2,j_2}\Big) \\ &\quad = \sum_{j_1,j_2,\cdots,j_k} c_{1,j_1} c_{2,j_2} \cdots c_{k,j_k} (u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,j_k})(f_{1,j_1}, f_{2,j_2}, \cdots, f_{k,j_k}) \\ &\quad = \sum_{j_1,j_2,\cdots,j_k} c_{1,j_1} c_{2,j_2} \cdots c_{k,j_k} \delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_k,j_k} = c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k}.\end{aligned}$$

It follows that the set of elements of the form $(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ spans $U_1 \otimes U_2 \otimes \cdots \otimes U_k$ because given $L \in U_1 \otimes U_2 \otimes \cdots \otimes U_k$, for $g_1, g_2, \cdots, g_k$ with each $g_i \in U_i^*$, with say $g_i = \sum c_{ij} f_{ij}$, we have

$$\begin{aligned}L(g_1, g_2, \cdots, g_k) &= L\Big(\sum_{i_1} c_{1,i_1} f_{1,i_1}, \sum_{i_2} c_{2,i_2} f_{2,i_2}, \cdots, \sum_{i_k} c_{k,i_k} f_{k,i_k}\Big) \\ &= \sum_{i_1,i_2,\cdots,i_k} c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k} L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}) \\ &= \sum_{i_1,i_2,\cdots,i_k} L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(g_1, g_2, \cdots, g_k)\end{aligned}$$

so $L = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ with $a_{i_1 i_2 \cdots i_k} = L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})$.

11.8 Example: For finite dimensional vector spaces U and V, there is a natural isomorphism $U^* \otimes V \cong \text{Lin}(U,V)$ obtained by identifying the element $f \otimes v \in U^* \otimes V$ with the linear map $f \otimes v : U \to V$ given by $(f \otimes v)(u) = f(u)v$.

11.9 Remark: For finite dimensional vector spaces $U_1, U_2, \cdots, U_k$ and V, there is a natural isomorphism between the space of k-linear maps $L : U_1 \times U_2 \times \cdots \times U_k \to V$ and the space of linear maps $M : U_1 \otimes U_2 \otimes \cdots \otimes U_k \to V$. This isomorphism sends the k-linear map $L : U_1 \times U_2 \times \ldots \times U_k \to V$ to the liner map $M : U_1 \otimes U_2 \otimes \cdots \otimes U_k \to V$ given by $M(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = L(u_1, u_2, \cdots, u_k)$ for all $u_i \in U_i$.

11.10 Remark: When some of the vector spaces $U_1, U_2, \cdots, U_k$ are infinite dimensional, for vectors $u_1, u_2, \cdots, u_k$ with each $u_i \in U_i$, we can still define the k-linear map

$$u_1 \otimes u_2 \otimes \cdots \otimes u_k : U_1^* \times U_2^* \times \cdots \times U_k^* \to \mathbb{F}$$

by

$$(u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) = g_1(u_1)g_2(u_2) \cdots g_k(u_k).$$

When $\mathcal{U}_i$ is a basis for U_i for each i, the set of k-linear maps

$$\mathcal{S} = \left\{ (u_1 \otimes u_2 \otimes \cdots \otimes u_k) \middle| \text{ each } u_i \in \mathcal{U}_i \right\}$$

is linearly independent (but does not span the vector space of all k-linear maps). In this case we define the tensor product $U_1 \otimes U_2 \otimes \cdots \otimes U_k$ to be the span of $\mathcal{S}$.

11.11 Example: We have natural isomorphisms $\mathbb{F}[x] \otimes \mathbb{F}[x] \cong \mathbb{F}[x] \otimes \mathbb{F}[y] \cong \mathbb{F}[x,y]$. The element $f(x) \otimes g(x) \in \mathbb{F}[x] \otimes \mathbb{F}[x]$ corresponds to the element $f(x) \otimes g(y) \in \mathbb{F}[x] \otimes \mathbb{F}[y]$ which corresponds to the element $f(x)g(y) \in \mathbb{F}[x,y]$.

11.12 Definition: For $k \in \mathbb{Z}^+$ we let S_k denote the set of all permutations of $\{1, 2, \cdots, k\}$, that is the set of all bijective maps $\sigma : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$. For a permutation $\sigma \in S_k$ we denote the **parity** of σ by $(-1)^\sigma$, in other words $(-1)^\sigma = \det(P_\sigma)$ where P_σ is the $k \times k$ permutation matrix $P_\sigma = (e_{\sigma(1)}, e_{\sigma(2)}, \cdots e_{\sigma(k)})$

11.13 Definition: Let U and V be vector spaces over a field $\mathbb{F}$. Let $L : U^k \to V$ be k-linear. We say that L is **symmetric** when

$$L(u_1, \cdots, u_i, \cdots, u_j, \cdots, u_k) = L(u_1, \cdots, u_j, \cdots, u_i, \cdots, u_k)$$

for all indices i, j and all vectors $u_1, u_2, \cdots, u_k \in U$. Equivalently L is symmetric when

$$L(u_1, u_2, \cdots, u_k) = L(u_{\sigma(1)}, u_{\sigma(2)}, \cdots, u_{\sigma(k)})$$

for all vectors $u_1, u_2, \cdots, u_k \in U$ and for every permutation $\sigma \in S_k$. We say that L is **alternating** (or **skew-symmetric**) when

$$L(u_1, \cdots, u_i, \cdots, u_j, \cdots, u_k) = -L(u_1, \cdots, u_j, \cdots, u_i, \cdots, u_k)$$

for all indices i, j and all vectors $u_1, u_2, \cdots, u_k \in U$. Equivalently, L is skew-symmetric when

$$L(u_1, u_2, \cdots, u_k) = (-1)^\sigma L(u_{\sigma(1)}, u_{\sigma(2)}, \cdots, u_{\sigma(k)})$$

for all vectors $u_1, u_2, \cdots, u_k \in U$ and all permutations $\sigma \in S_k$.

11.14 Definition: Let U be a finite dimensional vector space. We define the space of **k-tensors** on U, the space of **symmetric k-tensors** on U, and the space of **alternating k-tensors** on U to be

$$\begin{aligned}T^kU &= \bigotimes_{i=1}^{k} U = U\otimes U\otimes\cdots\otimes U = \big\{k\text{-linear maps } L:(U^*)^k\to\mathbb{F}\big\},\\ S^kU &= \big\{S\in T^kU\big|S \text{ is symmetric}\big\},\\ \Lambda^kU &= \big\{A\in T^kU\big|A \text{ is alternating}\big\}.\end{aligned}$$

11.15 Example: We have $T^1U=S^1U=\Lambda^1U=\big\{\text{linear maps } L:U^*\to\mathbb{F}\big\}=U^{**}$, which we identify with U.

11.16 Definition: Let U be a finite dimensional vector space. For $u_1,u_2,\cdots,u_k\in U$, we defined the **tensor product** $(u_1\otimes u_2\otimes\cdots\otimes u_k)\in T^kU$ by

$$(u_1\otimes u_2\otimes\cdots\otimes u_k)(g_1,g_2,\cdots,g_k)=g_1(u_1)g_2(u_2)\cdots g_k(u_k)$$

where each $g_i\in U^*$. We also define the **symmetric product** $u_1\odot u_2\odot\cdots\odot u_k\in S^kU$ by

$$\begin{aligned}(u_1\odot u_2\odot\cdots\odot u_k)(g_1,g_2,\cdots,g_k) &= \sum_{\sigma\in S_k}(u_1\otimes u_2\otimes\cdots\otimes u_k)(g_1,g_2,\cdots,g_k)\\ &= \sum_{\sigma\in S_k} g_{\sigma(1)}(u_1)g_{\sigma(2)}(u_2)\cdots g_{\sigma(k)}(u_k).\end{aligned}$$

and we define the **wedge product** $u_1\wedge u_2\wedge\cdots\wedge u_k\in\Lambda^kU$ by

$$\begin{aligned}(u_1\wedge u_2\wedge\cdots\wedge u_k)(g_1,g_2,\cdots,g_k) &= \sum_{\sigma\in S_k}(-1)^\sigma(u_1\otimes u_2\otimes\cdots\otimes u_k)(g_{\sigma(1)}g_{\sigma(2)}\cdots,g_{\sigma(k)})\\ &= \sum_{\sigma\in S_k}(-1)^\sigma g_{\sigma(1)}(u_1)g_{\sigma(2)}(u_2)\cdots g_{\sigma(k)}(u_k)\\ &= \det\begin{pmatrix} g_1(u_1) & g_1(u_2) & \cdots & g_1(u_k)\\ g_2(u_1) & g_2(u_2) & \cdots & g_2(u_k)\\ \vdots & \vdots & & \vdots\\ g_k(u_1) & g_k(u_2) & \cdots & g_k(u_k)\end{pmatrix}\end{aligned}$$

11.17 Theorem: *Let U be a finite dimensional vector space. Let $\mathcal{U}=\{u_1,u_2,\cdots,u_n\}$ be a basis for U. Then*

(1) $\big\{(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_k})\,\big|\,1\le i_1,i_2,\cdots,i_k\le n\big\}$ is a basis for T^kU,
(2) $\big\{(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_k})\,\big|\,1\le i_1\le i_2\le\cdots\le i_k\le n\big\}$ is a basis for S^kU, and
(3) $\big\{(u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_k})\,\big|\,1\le i_1<i_2<\cdots<i_k\le n\big\}$ is a basis for Λ^kU.

In particular we have $\dim\big(T^kU\big)=n^k$, $\dim\big(S^kU\big)=\binom{n+k-1}{k}$ *and* $\dim\big(\Lambda^kU\big)=\binom{n}{k}$.

Proof: Part (1) follows immediately from Theorem 11.7. We shall prove Part (3) and leave the proof of Part (2) as an exercise. Let $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ be the dual basis for U^*. Note that

$$\big(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}\big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) = \det \begin{pmatrix} f_{j_1}(u_{i_1}) & f_{j_1}(u_{i_2}) & \cdots & f_{j_1}(u_{i_k}) \\ \vdots & \vdots & & \vdots \\ f_{j_k}(u_{i_1}) & f_{j_k}(u_{i_2}) & \cdots & f_{j_k}(u_{i_k}) \end{pmatrix}$$

$$= \det \begin{pmatrix} \delta_{i_1,j_1} & \delta_{i_1,j_2} & \cdots & \delta_{i_1,j_k} \\ \vdots & \vdots & & \vdots \\ \delta_{i_1,j_k} & \delta_{i_2,j_k} & \cdots & \delta_{i_k,j_k} \end{pmatrix}$$

$$= \begin{cases} 0 & \text{if for some } l \text{ we have } i_l \neq j_m \text{ for all } m \\ 0 & \text{if } i_l = i_m \text{ for some } l \neq m \\ (-1)^\sigma & \text{if } i_l = j_{\sigma(l)} \text{ for all } l \text{ and some } \sigma \in S_k. \end{cases}$$

In particular, when $I = (i_1, i_2, \cdots, i_k)$ and $J = (j_1, j_2, \cdots, j_k)$ are increasing (that is when $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$) we have

$$\big(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}\big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) = \begin{cases} 0 \text{ if } I = J \\ 1 \text{ if } I \neq J. \end{cases}$$

It follows that the set

$$\mathcal{S} = \big\{u_I = (u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}) \big| I = (i_1, i_2, \cdots, i_k) \text{ is increasing}\big\}$$

is linearly independent because if $\sum\limits_{I \text{ incr}} a_I u_I = 0$ then for all increasing $J = (j_1, j_2, \cdots, j_k)$ we have

$$0 = \Big(\sum_{I \text{ incr}} a_I u_I\Big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) = a_J.$$

Given $L \in \Lambda^k U$, for each increasing $I = (i_1, i_2, \cdots, i_k)$, let $a_I = L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})$. Then for $g_1, g_2, \cdots, g_k \in U^*$ with say $g_j = \sum\limits_i c_{j,i} f_i$, we have

$$\begin{aligned} L(g_1, g_2, \cdots, g_k) &= L\Big(\sum_{i_1} c_{1,i_1} f_{i_1}, \sum_{i_2} c_{2,i_2} f_{i_2}, \cdots, \sum_{i_k} c_{k,i_k} f_{i_k}\Big) \\ &= \sum_{\text{all } I} \big(c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k}\big) L\big(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}\big) \\ &= \sum_{I \text{ incr}} \sum_{\sigma \in S_k} \big(c_{1,i_{\sigma(1)}} c_{2,i_{\sigma(2)}} \cdots c_{k,i_{\sigma(k)}}\big)(-1)^\sigma L\big(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}\big) \\ &= \sum_{I \text{ incr}} a_I \sum_{\sigma \in S_k} (-1)^\sigma c_{1,i_{\sigma(1)}} c_{2,i_\sigma(2)} \cdots c_{k,i_{\sigma(k)}} \\ &= \sum_{I \text{ incr}} a_I \det \begin{pmatrix} c_{1,i_1} & c_{1,i_2} & \cdots & c_{1,i_k} \\ \vdots & \vdots & & \vdots \\ c_{k,i_1} & c_{k,i_2} & \cdots & c_{k,i_k} \end{pmatrix} = \sum_{I \text{ incr}} a_I u_I(g_1, g_2, \cdots, g_k). \end{aligned}$$

Thus we have $L = \sum\limits_{I \text{ incr}} a_I u_I \in \text{Span}\,(\mathcal{S})$ and so $\mathcal{S}$ spans $\Lambda^k U$.

11.18 Example: Let $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$ and $\mathcal{V} = \{v_1 v_2, \cdots, v_n\}$ be two bases for U. Let $\alpha \in \Lambda^k U$. Say $\alpha = \sum\limits_{I \text{ incr}} a_I u_I = \sum\limits_{J \text{ incr}} b_J v_J$. Determine how a_I and b_J are related.

Solution: Let $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ and $\mathcal{G} = \{g_1, g_2, \cdots, g_n\}$ be the bases for U^* which are dual to $\mathcal{U}$ and $\mathcal{V}$. Let P be the change of basis matrix $P = [I]_{\mathcal{U}}^{\mathcal{V}}$ so that we have $v_j = \sum\limits_i p_{ij} u_i$. Note that

$$f_i(v_j) = f_i\big(\sum_k p_{kj} u_k\big) = \sum p_{kj} f_i(u_k) = \sum_k p_{kj} \delta_{ik} = p_{ij} \,.$$

We have

$$\begin{aligned} a_I = \alpha(f_{i_1}, f_{i_2}, \cdots, f_{i_k}) &= \sum_J b_J v_J(f_{i_1}, f_{i_2}, \cdots, f_{i_k}) \\ &= \sum_J b_J \det \begin{pmatrix} f_{i_1}(v_{j_1}) & f_{i_1}(v_{j_2}) & \cdots & f_{i_1}(v_{j_k}) \\ \vdots & \vdots & & \vdots \\ f_{i_k}(v_{j_1}) & f_{i_k}(v_{j_2}) & \cdots & f_{i_k}(v_{j_k}) \end{pmatrix} \\ &= \sum_J b_J \det \begin{pmatrix} p_{i_1,j_1} & p_{i_1,j_2} & \cdots & p_{i_1,j_k} \\ \vdots & \vdots & & \vdots \\ p_{i_k,j_1} & p_{i_k,j_2} & \cdots & p_{i_k,j_k} \end{pmatrix} \end{aligned}$$

11.19 Definition: Given an n-dimensional vector space U, we define vector spaces

$$TU = \bigoplus_{k=0}^{\infty} T^k U \ , \ SU = \bigoplus_{k=0}^{\infty} \ , \ \Lambda U = \bigoplus_{k=0}^{n} \Lambda^k U.$$

The operations $\otimes$, $\odot$ and $\wedge$, which are defined on basis vectors, determine products on each of the above vector spaces. A vector space with a compatible multiplication is called an algebra, so the above three vector spaces, together with their products, are called the **tensor algebra**, the **symmetric algebra**, and the **exterior algebra**.

11.20 Example: If $\alpha \in \Lambda^k U$ and $\beta \in \Lambda^l U$ then we have $\alpha \wedge \beta \in \Lambda^{k+l} U$. Indeed if $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$ is a basis for U and we have $\alpha = \sum\limits_{I \text{ incr}} a_I u_I$ and $\beta = \sum\limits_{J \text{ incr}} b_J u_J$, then

$$\alpha \wedge \beta = \sum_{I \text{ incr}} \sum_{J \text{ incr}} a_I b_J \ u_I \wedge u_J$$

where

$$\begin{aligned} u_I \wedge u_J &= (u_{i_1} \wedge \cdots \wedge u_{i_k}) \wedge (u_{j_1} \wedge \cdots \wedge u_{j_l}) \\ &= u_{i_1} \wedge \cdots \wedge u_{i_k} \wedge u_{j_1} \wedge \cdots \wedge u_{j_l}. \end{aligned}$$