MATH 239 Intro to Combinatorics, Solutions to Midterm Test, Winter 2012

[5] **1:** Given positive integers k, m and n with $m \leq n$, find the number of k-element subsets $A \subseteq \{1, 2, \cdots, n\}$ such that at least one element of A lies in $\{1, 2, \cdots, m\}$.

Solution 1: The number of k-element subsets of $\{1, 2, \cdots, n\}$ is equal to $\binom{n}{k}$. The k-element subsets of $\{1, 2, \cdots, n\}$ which contain no elements from $\{1, 2, \cdots, m\}$ are the same as the k-element subsets of $\{m+1, m+2, \cdots, n\}$, and the number of these is equal to $\binom{n-m}{k}$. Thus the number of k-element subsets of $\{1, 2, \cdots, n\}$ which contain at least one element from $\{1, 2, \cdots, m\}$ is equal to $\binom{n}{k} - \binom{n-m}{k}$.

Solution 2: The number of k-element subsets $A \subseteq \{1, 2, \cdots, n\}$ such that exactly i of the elements of A lie in $\{1, 2, \cdots, m\}$ is equal to $\binom{m}{i}\binom{n-m}{k-i}$, because there are $\binom{m}{i}$ ways to choose i elements from $\{1, 2, \cdots, m\}$ and there are $\binom{n-m}{k-i}$ ways to choose $k-i$ elements from $\{m+1, m+2, \cdots, n\}$. Thus the total number of k-element subsets of $\{1, 2, \cdots, n\}$ which contain at least one element from $\{1, 2, \cdots, m\}$ is equal to $\sum_{i \geq 1} \binom{m}{i}\binom{n-m}{k-i}$.

[5] **2:** Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $S = A^2$ with weight given by $w(a, b) = \gcd(a, b)$. Find the generating function $\phi_S(x)$.

Solution: We make a table showing $\gcd(a, b)$ for all possible pairs $(a, b) \in S$.

$a \backslash b$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

We see that there are 23 pairs (a, b) with $w(a, b) = 1$, 7 pairs with $w(a, b) = 2$, 3 pairs with $w(a, b) = 3$, 1 pair with $w(a, b) = 4$, 1 pair with $w(a, b) = 5$ and 1 pair with $w(a, b) = 6$ and so $\phi_S(x) = 23x + 7x^2 + 3x^3 + x^4 + x^5 + x^6$.

[5] **3:** Let S be a set with generating function $\phi_S(x) = \dfrac{1 + 2x}{1 - 2x - 3x^2}$. Find a closed-form formula for $|S_n|$.

Solution: Write $\phi_S(x) = \sum_{n \geq 0} c_n x^n$. Then we have

$$(1 - 2x - 3x^2)(c_0 + c_1 x + c_2 x^2 + \cdots) = 1 + 2x$$
$$c_0 + (c_1 - 2c_0)x + (c_2 - 2c_1 - 3c_0)x^2 + \cdots + (c_n - 2c_{n-1} - 3c_{n-2})x^n + \cdots = 1 + 2x$$

and so $c_0 = 1$, $c_1 = 4$, and $c_n - 2c_{n-1} - 3c_{n-2} = 0$ for $n \geq 2$. To solve this recursion, we let $g(x) = x^2 - 2x - 3$. Note that $g(x) = (x - 3)(x + 1)$, and hence $c_n = A \cdot 3^n + B(-1)^n$ for some constants A, B. To get $c_0 = 1$ and $c_1 = 4$ we need $A + B = 1$ and $3A - B = 4$, and so we must have $A = \frac{5}{4}$ and $B = -\frac{1}{4}$. Thus $c_n = \frac{5}{4} \cdot 3^n - \frac{1}{4}(-1)^n = \frac{5 \cdot 3^n - (-1)^n}{4}$.

[5] **4:** Let S be the set of binary strings which do not contain 0011 as a substring. Find the generating function (with respect to length) for S, expressed as a rational function.

Solution: Note that

$$S = \{\epsilon, 1, 11, 111, \cdots\}\big(\{01, 011, 0111, \cdots\} \cup \{001, 0001, 00001, \cdots\}\big)^* \{\epsilon, 0, 00, 000, \cdots\}.$$

We have

$$\begin{aligned}
\phi_{\{\epsilon,1,11,111,\cdots\}} = \phi_{\{\epsilon,0,00,000,\cdots\}} &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\
\phi_{\{01,011,0111,\cdots\}\cup\{001,0001,00001,\cdots\}} &= (x^2 + x^3 + x^4 + \cdots) + (x^3 + x^4 + x^5 + \cdots) \\
&= \frac{x^2}{1-x} + \frac{x^3}{1-x} = \frac{x^2+x^3}{1-x} \\
\phi_S = \left(\frac{1}{1-x}\right)^2 \frac{1}{1 - \frac{x^2+x^3}{1-x}} &= \frac{1}{(1-2x+x^2) - (x^2 - x^4)} = \frac{1}{1-2x+x^4}.
\end{aligned}$$

[5] **5:** For each positive integer n, let c_n be the number of integer sequences $(a_0, a_1, \cdots, a_n)$ with $a_0 = a_n = 0$ and $\left|a_i - a_{i-1}\right| \le 2$ for $i = 1, 2, \cdots, n$. Find a formula for c_n expressed as a sum, and in particular find c_5. One way to solve this problem is to use the bijection

$$(a_0, a_1, \cdots, a_n) \mapsto (a_1 - a_0 + 2, a_2 - a_1 + 2, \cdots, a_n - a_{n-1} + 2) = (b_1, b_2, \cdots, b_n).$$

Solution: From the above bijection, we see that the number of such sequences $(a_0, \cdots, a_n)$ is equal to the number of sequences $(b_1, b_2, \cdots, b_n)$ with each $b_i \in \{0, 1, 2, 3, 4\}$ and with $\sum b_i = 2n$. Let S be the set of sequences $(b_1, b_2, \cdots, b_n)$ with each $b_i \in \{0, 1, 2, 3, 4\}$ (equivalently, let $S = \{0, 1, 2, 3, 4\}^n$) with weight $w(b_1, \cdots, b_n) = \sum b_i$. We need to find $|S_{2n}| = \left[x^{2n}\right]\phi_S(x)$. We have

$$\begin{aligned}
\phi_{\{0,1,2,3,4\}}(x) &= 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x} \\
\phi_S(x) &= \left(\frac{1-x^5}{1-x}\right)^n = (1-x^5)^n(1-x)^{-n} \\
&= \sum_{i\ge 0} (-1)^i \binom{n}{i} x^{5i} \sum_{j \ge 0} \binom{n+j-1}{n-1} x^j
\end{aligned}$$

To find $\left[x^{2n}\right]$ we choose the term with $j = 2n - 5i$ so that $5i + j = 2n$, and we obtain

$$|S_{2n}| = \left[x^{2n}\right]\phi_S(x) = \sum_{i \ge 0} (-1)^i \binom{n}{i} \binom{3n-5i-1}{n-1}.$$

In particular, taking $n = 5$ gives

$$|S_{10}| = \tbinom{5}{0}\tbinom{14}{4} - \tbinom{5}{1}\tbinom{9}{4} + \tbinom{5}{2}\tbinom{4}{4} = 1 \cdot 1001 - 5 \cdot 126 + 10 \cdot 1 = 381.$$

[5] **6:** For each positive integer n, let c_n be the number of strings $(x_1, x_2, \cdots, x_n)$, with each $x_i \in \{1, 2, 3\}$, which do not contain either 12 or 21 as substrings. Find a recursion formula for c_n, and in particular, find c_5. One way to solve this problem is to let a_n be the number of such sequences which end with 1 (which is equal to the number of such sequences which end with 2) and let b_n be the number of such sequences which end with 3, then use relationships between the numbers a_i and b_i to find the recursion formula for c_n.

Solution: Let a_n and b_n be as suggested. Note that (1) $c_n = 2a_n + b_n$ for all $n \geq 1$ (since every allowable string ends with 1, 2 or 3), (2) $a_n = a_{n-1} + b_{n-1}$ for all $n \geq 2$ (an allowable string of length n which ends with 1 can be obtained by appending a 1 to the end of an allowable string of length $n-1$ which ends with 1 or with 3) and (3) $b_n = c_{n-1}$ for all $n \geq 2$ (an allowable string of length n which ends with 3 can be obtained by appending a 3 to the end of an allowable string of length $n-1$). Substituting $b_n = c_{n-1}$ into equation (1) gives $c_2 = 2a_n + c_{n-1}$, so we have $a_n = \frac{1}{2}\big(c_n - c_{n-1}\big)$ for all $n \geq 2$. Substituting $a_n = \frac{1}{2}(c_n - c_{n-1})$, $a_{n-1} = \frac{1}{2}(c_{n-1} - c_{n-2})$ and $b_{n-1} = c_{n-2}$ into equation (2) gives $\frac{1}{2}(c_n - c_{n-1}) = \frac{1}{2}(c_{n-1} - c_{n-2}) + c_{n-2}$, and so we have $c_n = 2c_{n-1} + c_{n-2}$ for all $n \geq 3$. Finally note the $c_1 = 3$ and $c_2 = 7$. Using this recursion formula, the first few terms c_n are as follows

n	1	2	3	4	5
c_n	3	7	17	41	99