Name (print): ____________________

Signature: ____________________

ID Number: ____________________

MATH 239, Introduction to Combinatorics

Midterm Test, Winter Term, 2012

University of Waterloo, UAE Campus

Instructor: Stephen New

Date: February 26, 2012

Time: 12:00 – 1:30 pm

Question	Mark
1	/5
2	/5
3	/5
4	/5
5	/5
6	/5
Total	/30

Instructions:

1. Place your name, signature and ID number in the spaces provided at the top of this page.

2. This test contains 7 pages, including this cover page and a page at the end for rough work.

3. No calculators are allowed.

4. Answer all 6 questions; all questions will be given equal value.

5. Provide full explanations with all your solutions.

[5] **1:** Given positive integers k, m and n with $m \leq n$, find the number of k-element subsets $A \subseteq \{1, 2, \cdots, n\}$ such that at least one element of A lies in $\{1, 2, \cdots, m\}$.

[5] **2:** Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $S = A^2$ with weight given by $w(a, b) = \gcd(a, b)$. Find the generating function $\phi_S(x)$.

[5] **3:** Let S be a set with generating function $\phi_S(x) = \dfrac{1+2x}{1-2x-3x^2}$. Find a closed-form formula for $|S_n|$.

[5] **4:** Let S be the set of binary strings which do not contain 0011 as a substring. Find the generating function (with respect to length) for S, expressed as a rational function.

[5] **5:** For each positive integer n, let c_n be the number of integer sequences $(a_0, a_1, \cdots, a_n)$ with $a_0 = a_n = 0$ and $\left|a_i - a_{i-1}\right| \leq 2$ for $i = 1, 2, \cdots, n$. Find a formula for c_n expressed as a sum, and in particular find c_5. One way to solve this problem is to use the bijection

$$(a_0, a_1, \cdots, a_n) \mapsto (a_1 - a_0 + 2, a_2 - a_1 + 2, \cdots, a_n - a_{n-1} + 2) = (b_1, b_2, \cdots, b_n)\,.$$

[5] **6:** For each positive integer n, let c_n be the number of strings $(x_1, x_2, \cdots, x_n)$, with each $x_i \in \{1, 2, 3\}$, which do not contain either 12 or 21 as substrings. Find a recursion formula for c_n, and in particular, find c_5. One way to solve this problem is to let a_n be the number of such sequences which end with 1 (which is equal to the number of such sequences which end with 2) and let b_n be the number of such sequences which end with 3, then use relationships between the numbers a_i and b_i to find the recursion formula for c_n.

This page is for rough work. It will not be marked.