Name (print): _____

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MATH 239, Introduction to Combinatorics

Final Examination, Winter Term, 2012

University of Waterloo, UAE Campus

Instructor: Stephen New

Date: April 15, 2012

Time: 9:30 am - 12:00 noon

Instructions:

1. Place your name, signature and ID number in the spaces provided at the top of this page.

2. This test contains 8 pages, including this cover page and a page at the end for rough work.

3. No calculators are allowed.

4. Answer all 6 questions; all questions will be given equal value.

5. Provide full explanations with all your solutions.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

[10] **1:** (a) Given a positive integer n, find the number of sequences $(a_1, a_2, \cdots, a_n)$ with each $a_i \in \{1, 2, 3, 4\}$ such that no even number is followed by an odd number.

(b) Given positive integers k and n with $\frac{n}{2} \leq k \leq n$, find the number of ordered pairs (A, B) where A and B are k-element subsets of $S = \{1, 2, \cdots, n\}$ with $A \cup B = S$.

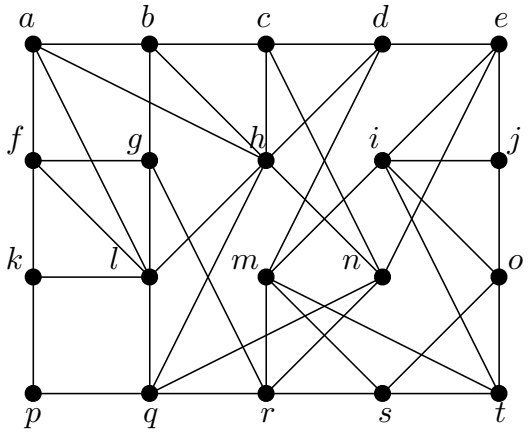
[10] **2:** (a) Let $\phi(x) = \dfrac{1+x+2x^2}{1-x+x^2-2x^3}$. Find $[x^8]\phi(x)$ (hint: use a recursion formula).

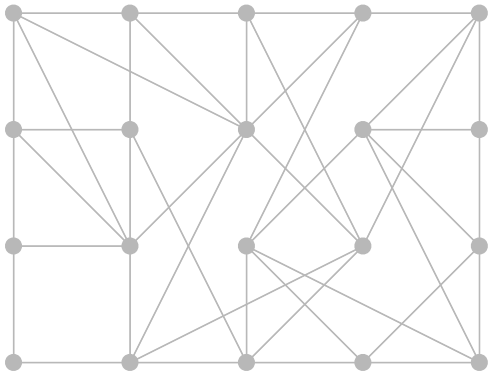
(b) Let $a_0 = 0$, $a_1 = 1$, $a_2 = 2$ and $a_n = 3a_{n-1} - 4a_{n-3}$ for $n \geq 3$. Find a non-recursive formula for a_n in terms of n $\big($hint: $x^3 - 3x^2 + 4 = (x-2)^2(x+1)\big)$.

[10] **3:** (a) Given positive integers n and k, find the number of compositions of n into $2k$ parts a_i with $a_i \equiv i \pmod 2$ for all i (hint: use a generating function).

(b) Find the generating function with respect to length, expressed as a rational function, for the set of binary strings in which no 0-block is followed by a 1-block of the same length.

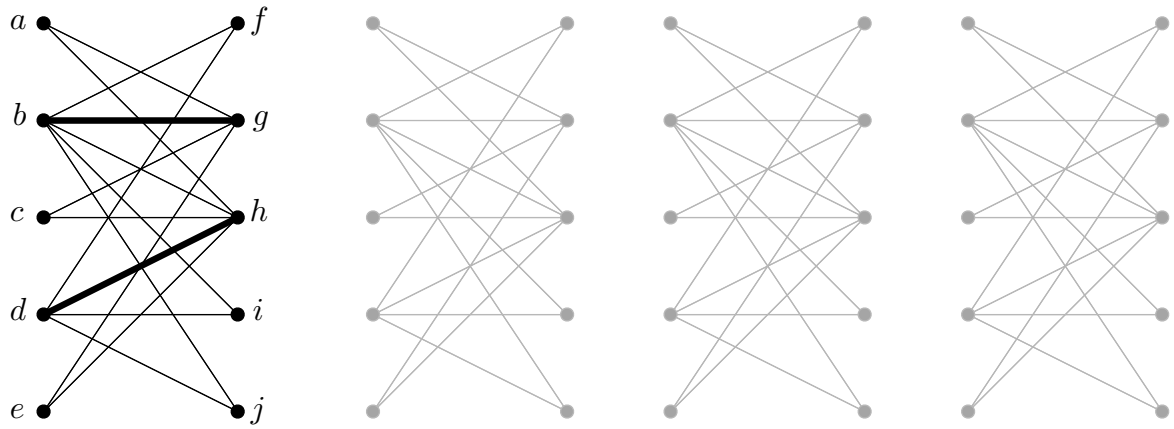
[10] **4:** (a) Find a BFS spanning tree in the following graph G, and use the tree to find a minimum length path in G from a to t. Show your work on the second copy of the graph, labelling the vertices $a_0, a_1, a_2, \cdots$ in the order in which they are added to the tree.



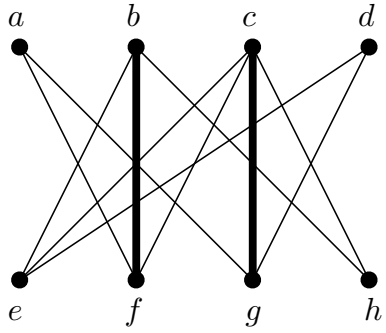


(b) Find the number of BFS spanning trees in the 3-cube Q_3 with root $a_0 = (0, 0, 0)$.

[10] **5:** (a) Use the Maximum Matching Algorithm to find a maximum matching and a minimum cover in the following graph G, beginning with the given matching (shown in bold). Show your work on the extra copies of the graph. At each stage, indicate the augmenting path.



(b) Find the total number of augmenting paths from some point $x \in A \setminus V(M)$ to some point $y \in B \setminus V(M)$ in the following graph, where $A = \{a, b, c, d\}$, $B = \{e, f, g, h\}$ and M is the given matching, shown in bold.



[10] **6:** (a) Let G be a graph with $\deg(v) \geq 3$ for every vertex v, which has a planar embedding in which $\deg(f) \geq 5$ for every face f. Show that G has at least 20 vertices.

(b) Find the number of connected graphs with vertex set $V = \{(0,0),(1,0),(0,1),(1,1)\}$ (hint: it is easier to count the disconnected graphs).

This page is for rough work. It will not be marked.