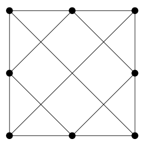
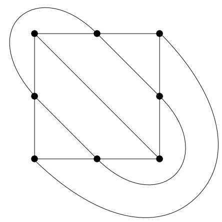
MATH 239 Intro to Combinatorics, Solutions to Assignment 6

1: Determine which of the following graphs are planar.

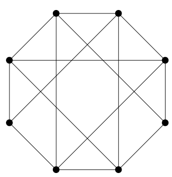
(a)



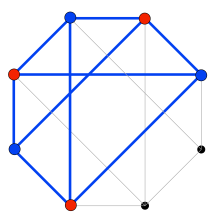
Solution: This graph is planar since we can redraw it as shown below.



(b)

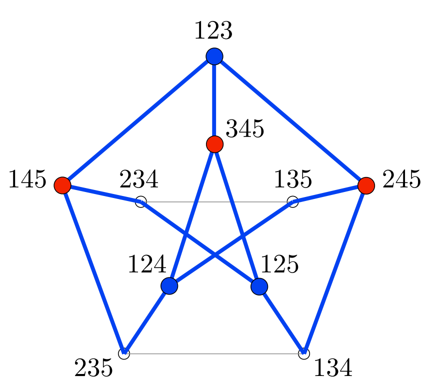


Solution: This graph is not planar since it contains a subgraph, as shown below, which is isomorphic to $K_{3,3}$.



(c) $S_{5,3,1}$

Solution: Note that $S_{5,3,1}$ is isomorphic to $S_{5,2,0}$ (the Peterson graph) under the map which sends the 3-element subset to its 2-element complement. This observation helps to draw a symmetric picture of the graph, as shown below. We see that it is not planar since it contains a subgraph, as indicated in the picture, which is isomorphic to an edge subdivision of $K_{3,3}$.

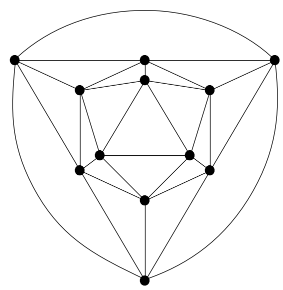


2: (a) Let G be a graph in which $\deg(v) \geq 5$ for every vertex v. Suppose G has a planar embedding such that $\deg(f) \geq 3$ for every face f. Show that G has at least 12 vertices. Find a planar embedding of such a graph with exactly 12 vertices.

Solution: We have $2q = \sum_{v \in V} \deg(v) \geq 5p$ and we have

$$2q = \sum_{f \in F} \deg(f) \geq 3r = 3(q - p + c + 1) = 3q - 3p + 3(c+1) \geq 3q - 3p + 6$$

so that $q \leq 3p - 6$. Combining these two inequalities gives $5p \leq 2q \leq 6p - 12$ and so $p \geq 12$, as required. The picture below shows a planar embedding of the regular icosahedron, which satisfies $p = 12$, $\deg(v) = 5$ for all vertices v, and $\deg(f) = 3$ for all faces f.



(b) Let G be a connected graph which has a planar embedding in which $\deg(f) \geq 4$ for every face f. Show that G has at least 3 vertices of degree at most 3. Find a planar embedding of such a graph with exactly 3 vertices of degree at most 3.

Solution: Note that $2q = \sum_{f \in F} \deg(f) \geq 4r$, so we have $q \geq 2r = 2q - 2p + 2(c+1) = 2q - 2p + 4$ and hence

$$q \leq 2p - 4 \quad (*)$$

If G had no vertices of degree at most 3, so that $\deg(v) \geq 4$ for all v, then we would have $2q = \sum \deg(v) \geq 4p$ so that $q \geq 2p$. But from (*) this would imply that $2p \leq q \leq 2p - 4$ so that $0 \leq -4$, which is not true. If G had exactly one vertex, say u, of degree at most 3, so that $\deg(v) \geq 4$ for all $v \neq u$, then we would have $2q = \deg(u) + \sum_{v \neq u} \deg(v) \geq 1 + 4(p-1) = 4p - 3$. But from (*) this would imply that $4p - 3 \leq 2q \leq 4p - 8$ so that $-3 \leq -8$, which is false. If G had exactly 2 vertices, say u_1 and u_2, of degree at most 3, so that $\deg(v) \geq 4$ for all $v \neq u_1, u_2$, then we would have $2q = \deg(u_1) + \deg(u_2) + \sum_{v \neq u_1, u_2} \deg(v) \geq 1 + 1 + 4(p-2) = 4p - 6$. But from (8) this would imply that $4p - 6 \leq 2q \leq 4p - 8$ so that $-6 \leq -8$, which is not true. Thus G must have at least 3 vertices of degree at most 3. The (unique) tree on 3 vertices, embedded in the plane as shown below, has two vertices of degree 1 and one of degree 2, and it has one face, which is of degree 4.