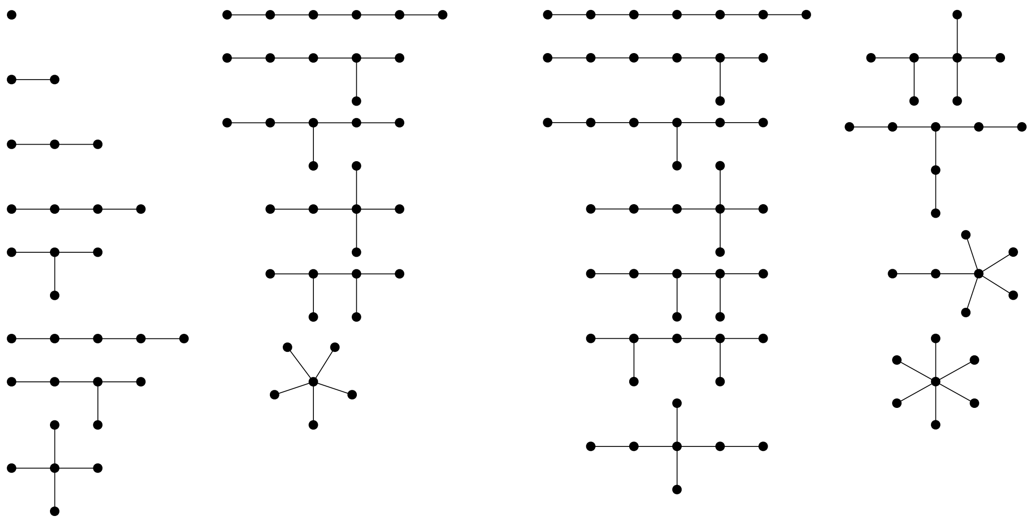
MATH 239 Intro to Combinatorics, Solutions to Assignment 5

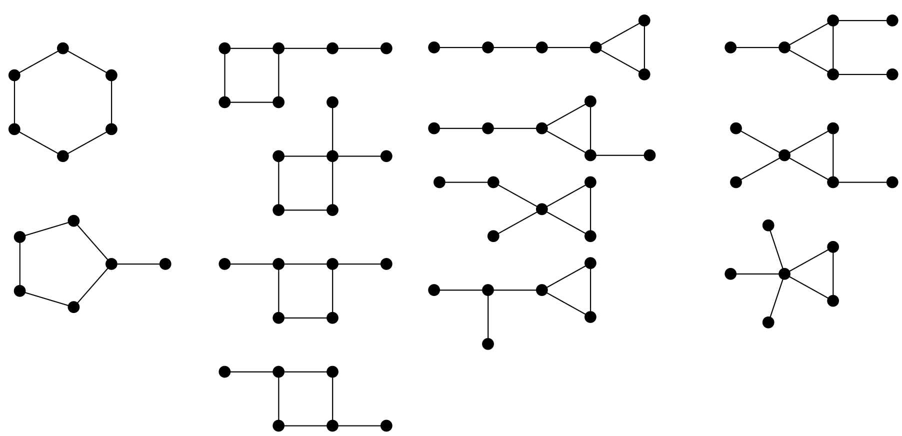
1: (a) Up to isomorphism, there are 25 trees with at most 7 vertices. Without proof, draw a picture of one tree from each of these 25 isomorphism classes.

Solution:



(b) Without proof, find the number of connected graphs G with 6 vertices and 6 edges, up to isomorphism. Draw a picture of one graph from each isomorphism class.

Solution: Each connected graph with 6 vertices and 6 edges has a spanning tree with 6 vertices and 5 edges, so we can obtain each such graph by adding one edge between two vertices in one of the trees from part (a).



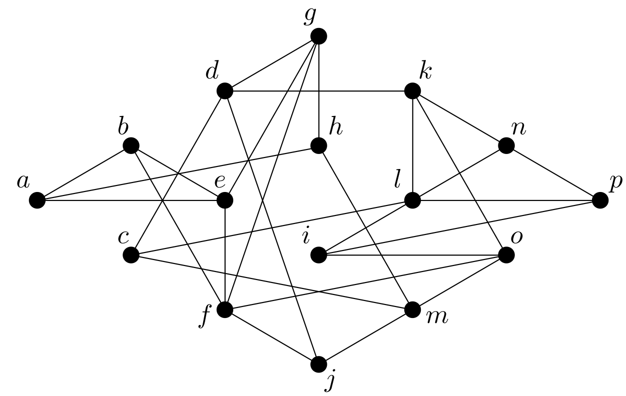
2: (a) Show that in any tree whose vertices all have odd degree, the number of vertices of degree 1 is greater than the number of vertices whose degree is not equal to 1.

Solution: Let T be a tree whose vertices all have odd degree. Say T has p vertices and q edges. Let A be the set of vertices of degree 1 and let B be the set of vertices of degree $3, 5, 7, \cdot$. We have $|A| + |B| = p$, $p = q + 1$ and $2q = \sum_{v \in V(G)} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) \geq |A| + 3|B|$. Thus $|A| + 3|B| \leq 2q = 2p - 2 = 2|A| + 2|B| - 2$ and so $|A| \geq |B| + 2$ hence $|A| > |B|$ as required.

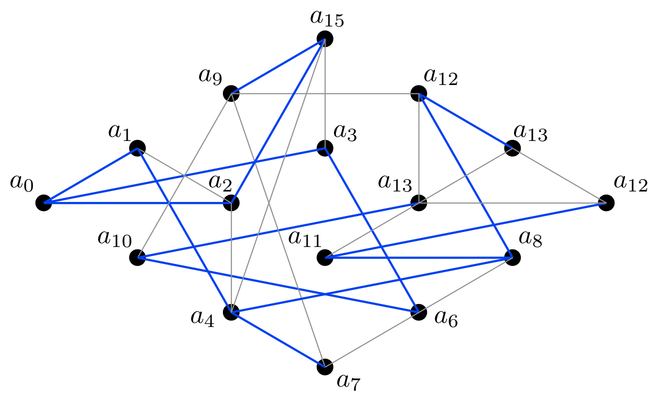
(b) Show that given any positive integers $d_1, d_2, \cdots, d_p$ with $d_1 + d_2 + \cdots + d_p = 2p - 2$, there exists a tree with p vertices $v_1, v_2, \cdots, v_p$ with $\deg(v_i) = d_i$ for all i.

Solution: We prove this by induction on p. When $p = 1$, to get $d_1 = 2p - 2 = 0$ we must take $d_1 = 0$ and then we can use the tree with 1 vertex and no edges. When $p = 2$, to get $d_1 + d_2 = 2p - 2 = 2$ we must take $d_1 = d_2 = 1$, and then we can use the tree with 2 vertices and 1 edge. Let $p \geq 3$ and suppose, inductively, that for all positive integers $c_1, c_2, \cdots, c_{p-1}$ with $\sum c_i = 2p - 4$ there exists a tree with $p - 1$ vertices $v_1, v_2, \cdots, v_{p-1}$ such that $\deg(v_i) = c_i$ for all i. Let $d_1, d_2, \cdots, d_p$ be positive integers with $\sum d_i = 2p - 2$. Note that we must have $d_i = 1$ for some i since if $d_i \geq 2$ for all i then $\sum d_i \geq 2p > 2p - 2$. Note that we must have $d_i \geq 2$ for some i since if $d_i = 1$ for all i then $\sum d_i = p < 2p - 2$ (since $p \geq 3$). Say $d_{p-1} \geq 2$ and $d_p = 1$. Let $c_i = d_i$ for $i < p - 1$ and let $c_{p-1} = d_{p-1} - 1$. Then $c_1, c_2, \cdots, c_{p-1}$ are positive integers with $\sum c_i = 2p - 4$, so by the induction hypothesis we can find a tree T with $p - 1$ vertices $v_1, v_2, \cdots, v_{p-1}$ with $\deg(v_i) = c_i$ for all i. Introduce a new vertex v_p and let $e = \{v_{p-1}, v_p\}$. Then (as in the proof of the BFS Tree Theorem), $T \cup [e$ is also a tree. The tree $T \cup e$ has p vertices $v_1, v_2, \cdots, v_p$ with $\deg(v_i) = d_i$ for all i.

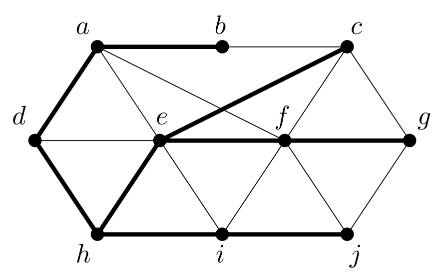
3: (a) Find a breadth-first search spanning tree, rooted at vertex a, in the graph shown below and use your tree to find a shortest path from a to p.



Solution: A BFS tree is shown below in blue. The vertices have been relabelled as $a_0, a_1, \cdots, a_{13}$ in the order in which they were added to the tree. The unique path in this tree from a to p is the path a, b, f, o, i, p. This is a shortest path from a to p in G. (Another shortest path is the path a, h, m, c, l, p).



(b) The picture below shows a spanning tree T (with edges in bold face) in a graph G. Determine whether it is possible to obtain T using a breadth-first search in G and, if so, determine the order in which the vertices were added to the tree.

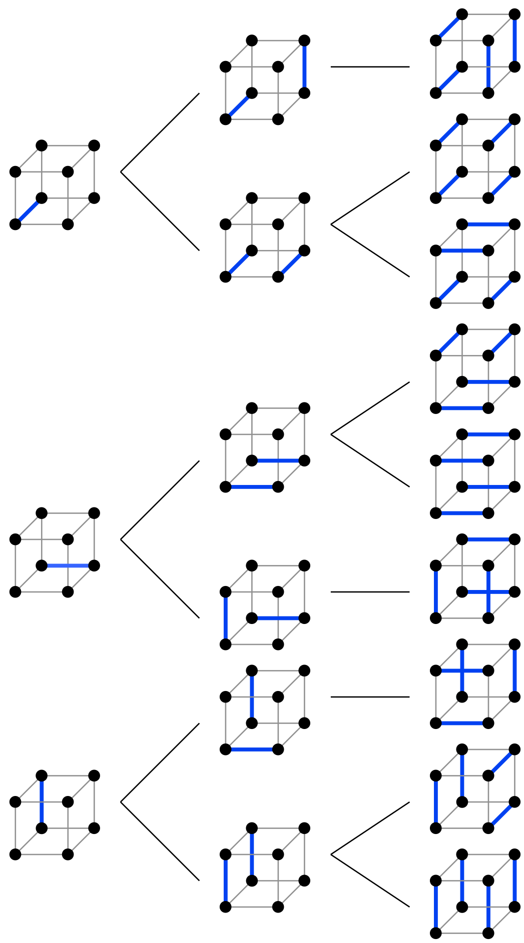


Solution: It is indeed possible to obtain this tree T using a BFS. We must select $a_0 = h$ since h is the only vertex of G at which every edge was selected to be in T. The next three vertices a_1, a_2, a_3 must be the endpoints of theses three edges so we have $\{a_1, a_2, a_3\} = \{d, e, i\}$. We cannot have $a_1 = i$ (since if so then we would have selected edge ie before ef to be in T) and we cannot have $a_1 = e$ (since if so then we would have selected edge ea before da to be in T) and so we must have $a_1 = d$. We cannot have $a_2 = i$ (since if so then edge if would have been selected before ef) so we must have $a_2 = e$. Since $a_1 = d$ and $a_2 = e$ we must have $a_3 = i$. Similar arguments show that the only possible choice for the vertices $a_0, a_1, \cdots, a_9$ is as follows.

i	0	1	2	3	4	5	6	7	8	9
a_i	h	d	e	i	a	f	c	j	b	g

4: (a) Find the number of perfect matchings in the 3-cube Q_3.

Solution: There are 9 perfect mathchings in Q_3. They can be listed in a systematic way with a tree diagram, such as the following.



(b) Find the number of perfect matchings in the complete bipartite graph $K_{n,n}$.

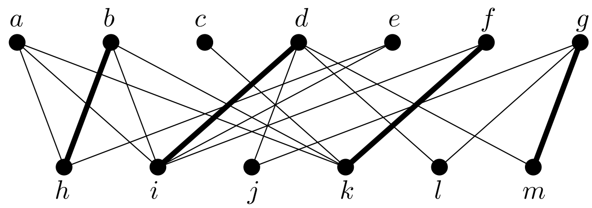
Solution: Say $K_{n,n}$ has bipartition A, B where $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$. To choose a perfect matching in $K_{n,n}$, here are n choices for the edge at a_1 (it could be $a_1 b_i$ for any i). Having chosen the edge at a_1, say edge $a_1 b_{k_1}$, there are $n- = 1$ choices for the edge at a_2 (it could be any edge $a_2 b_i$ with $i \neq k_1$). Having chosen the edge at a_2, say edge $a_2 b_{k_2}$, there are then $n - 2$ choices for the edge at a_3 (it could be any edge $a_3 b_i$ with $i \neq k_1, k_2$). Similarly there are $n - -3$ choices for the edge at a_4 and so on. Thus there are a total of $n(n-1)(n-2)\cdots(2) = n!$ perfect matchings.

(c) Find the number of perfect matchings in the complete graph K_{2n}.

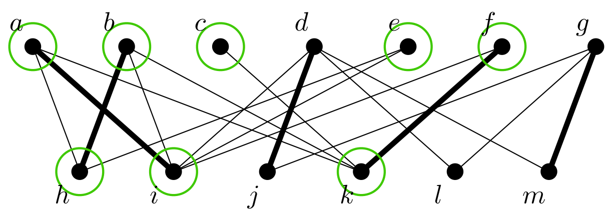
Solution: Let c_n be the number of perfect matchings in K_{2n}. Say the vertices of K_{2n} are $a_1, a_2, \cdots, a_{2n}$. When choosing a perfect matching in K_{2n}, there are $2n - 1$ choices for the edge at a_1 (it cowld be any edge $a_1 a_i$ with $i \neq 1$). Having chosen this edge, say $a_1 a_k$, if we remove the verices a_1 and a_k from the vertex set and if we remove all of the edges in K_n which touch either of these two vertices, then we are left with the complete graph $K_{2(n-1)}$ with vertices a_i, $i \neq 1, k$. There are c_{n-1} ways to choose a perfect matching in $K_{2(n-1)}$, and so the number of ways to choose a perfect matching in K_{2n} is equal to $c_n = (2n-1)c_{n-1}$. Since $c_1 = 1$ and $c_n = (2n-1)c_{n-1}$ we have $c_1 = 1$, $c_2 = 1 \cdots 3$, $c_3 = 1 \cdot 3 \cdot 5$ and so on, so $c_n = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) = \frac{(2n)!}{2^n\, n!}$.

5: For each of the following two graphs, use the bipartite matching algorithm, starting with the matching given by the bold edges, to find a maximum matching and a minimum cover.

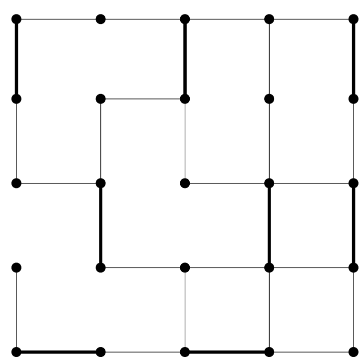
(a)



Solution: We use the augmenting path a, i, d, j to augment the given matching, replacing the edge id by the edges ai and dj. The resulting matching $M = \{ai, bh, dj, fk, gm\}$ is shown below. The vertices in the sets X and Y are circled in green. By systematically trying all possibilities, we find that there are no more augmenting paths, so the matching is maximum. A minimum cover is given by $C = (A \setminus X) \cup Y = \{d, g\} \cup \{h, i, k\} = \{d, g, h, i, k\}$. Notice that $|M| = |C| = 5$.



(b)



Solution: In the following sequence of pictures, we begin with the given matching with one edge added, and at each stage, we use the augmenting path shown in red to obtain the matching in the following picture. The matching in the final picture is clearly a maximum matching since it touches all vertices but 1. The cover C that is obtained from the bipartite matching algorithm depends on whether we choose A to consist of the white vertices or the black vertices. We choose A to consist of the white vertices and, in the final picture, we circle the vertices in X in grey, the vertices in $A \setminus X$ in green (there are 2 of these), and the vertices in Y in blue (there are 10 of these). The cover C consists of the green vertices and the blue vertices.