MATH 239 Intro to Combinatorics, Solutions to Assignment 4

1: (a) Find the incidence matrix for the graph with vertex set and edge set

$$V = \{1,2,3,4,5,6\} \text{ and } E = \big\{\{1,2\},\{1,3\},\{1,4\},\{1,6\},\{3,5\},\{4,5\},\{4,6\}\big\}.$$

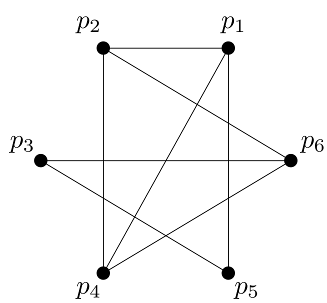
Solution: The incidence matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Using the 6 points in the plane given by $p_k = e^{i\,k\pi/3}$ for $k = 1, 2, \cdots, 6$ (these are the vertices of a regular hexagon), draw a picture of the graph with adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Solution:



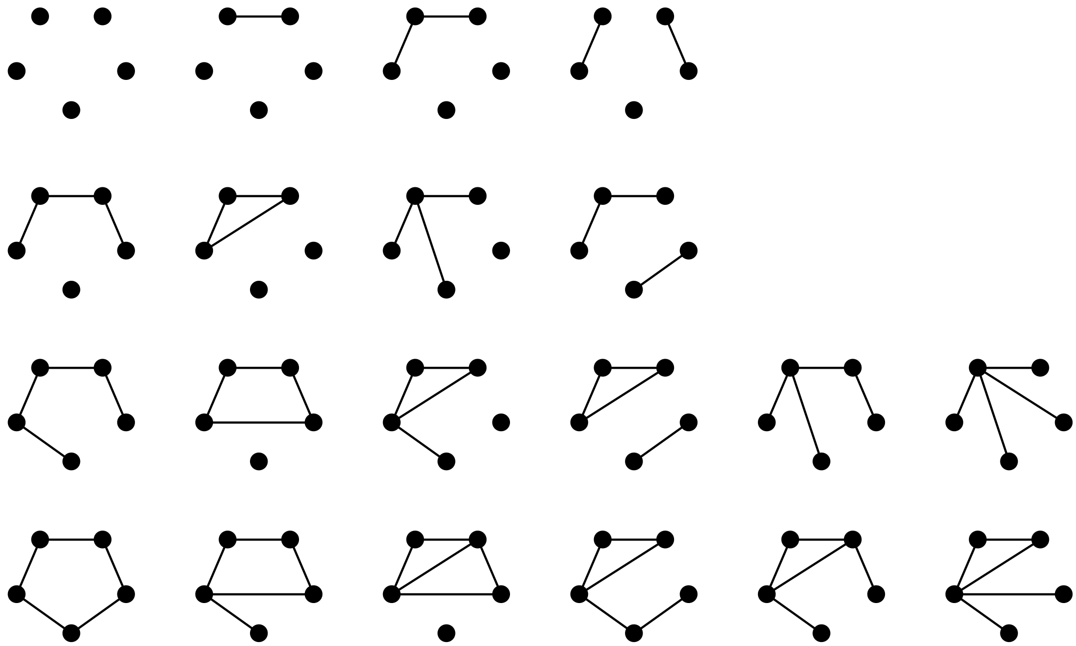
(c) Let A be the adjacency matrix for a graph G. Show that $(A^n)_{k,l}$ is equal to the number of walks of length n from v_k to v_l in G.

Solution: Consider the case that $n = 1$. A walk of length 1 is of the form u, v where $\{u, v\}$ is an edge. Thus if $A_{k,l} = 1$ (so $\{a_k, a_l\}$ is an edge) then there is exactly one walk of length 1 from a_k to a_l (namely the path a_k, a_l) and if $A_{k,l} = 0$ (so $\{a_k, a_l\}$ is not an edge) then there is no walk from a_k to a_l. Thus the number of walks of length 1 from a_k to a_l is equal to $A_{k,l}$. Now let $n \geq 1$ and suppose, inductively, that for all i, j, $(A^{n-1})_{i,j}$ is equal to the number of walks of length $n-1$ from a_i to a_j. Note that there is a bijective correspondence between the set of all walks of length n from a_k to a_l and the set of all walks of length $n-1$ from a_k to some vertex a_i which is joined by an edge to a_l (the walk $a_k = v_0, v_1, \cdots, v_n = a_l$ corresponds to the walk $a_k = v_0, v_1, \cdots, v_{n-1} = a_i$ with $\{v_{n-1}, v_n\} = \{a_i, a_k\}$). Thus

$$\begin{aligned} &\text{the \# of walks of length } n-1 \text{ from } a_k \text{ to } a_l \\ &= \sum_{i \text{ s.t. } \{v_i, v_k\} \in E} \big(\text{\# of walks of length } n-1 \text{ from } a_k \text{ to } a_i\big) \\ &= \sum_{i \text{ s.t. } \{v_i, v_k\} \in E} (A^{n-1})_{k,i} \text{ , by the induction hypothesis} \\ &= \sum_{i=1}^{p} (A^{n-1})_{k,i} A_{i,l} \text{ , since } A_{k,l} = \begin{cases} 1 \text{ if } \{a_k, a_i\} \in E \\ 0 \text{ if } \{a_k, a_i\} \notin E \end{cases} \\ &= (A^n)_{k,l} \text{ , by matrix multiplication.} \end{aligned}$$

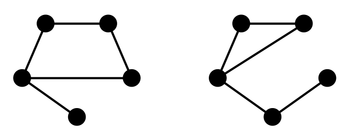
2: (a) Up to isomorphism, there are 20 graphs with 5 vertices with at most 5 edges. Without proof, draw a picture of one graph for each of these 20 isomorphism classes.

Solution:



(b) Find two connected graphs with 5 vertices which are not isomorphic but have the same number of vertices of each degree.

Solution:



3: (a) Let $S_{n,k,i}$ be the graph whose vertex set V is the set of k-element subsets of $\{1, 2, \cdots, n\}$ and whose edge set E is the set of 2-element sets $\{A, B\}$ with $A, B \in V$ such that $|A \cap B| = i$. Find the number of edges in $S_{n,k,i}$ and find the value of r such that $S_{n,k,i}$ is r-regular.

Solution: The number of edges in $S_{n,k,i}$ is equal to the number of unordered pairs $\{A, B\}$ where A and B are k-element subsets of $\{1, 2, \cdots, n\}$ with $|A \cap B| = i$. The number of ways to choose the k elements in A is equal to $\binom{n}{k}$. The number of ways to choose the i elements in $A \cap B$ from the k elements in A is equal to $\binom{k}{i}$. The number of ways to choose the remaining $k - i$ elements in B from the remaining $n - k$ elements in $\{1, 2, \cdots, n\} \setminus A$ is $\binom{n-k}{k-i}$. Thus the number of ways to choose the ordered pair (A, B) is equal to $\binom{n}{k}\binom{k}{i}\binom{n-k}{k-i}$ and so the number of edges is $\frac{1}{2}\binom{n}{k}\binom{k}{i}\binom{n-k}{k-i}$.

The required value of r is equal to the number of ways of choosing B when A is fixed. Having fixed the k-element set A, there are $\binom{k}{i}$ ways to choose the i elements in $A \cap B$ from A, and there are $\binom{n-k}{k-i}$ ways to choose the remaining $k - i$ elements from the set $\{1, 2, \cdots, n\} \setminus A$, and so $r = \binom{k}{i}\binom{n-k}{k-i}$.

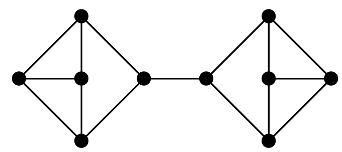
(b) Let K_n be the complete graph with vertex set $V = \{1, 2, \cdots, n\}$ whose edge set E is the set of all 2-element sets $\{a, b\}$ with $a, b \in V$. Find the number of subgraphs of K_n and find the number of paths from 1 to n in K_n.

Solution: To form a subgraph H of K_n, the vertex set of H can be any subset $S \subseteq V$ and then the edge set of H can be any subset of the set of all edges of K_n with both ends in S. For each k, there are $\binom{n}{k}$ ways to choose the k-element set S. The edges of K_n with both ends in S are precisely the 2-element subsets of S, and there are $\binom{k}{2} = \frac{k(k-1)}{2}$ of these. The number of subsets of this $\frac{k(k-1)}{2}$-element set is equal to $2^{k(k-1)/2}$. Thus the number of subgraphs H of K_n is equal to $\sum_{k=1}^{n} \binom{n}{k} 2^{k(k-1)/2}$.

A path of length l from 1 to n in K_n is of the form $1 = v_0, v_1, \cdots, v_l = n$ where the vertices v_i are distinct. There are $n - 2$ ways to choose v_1, then $n - 3$ ways to choose v_2, and so on, and so the number of paths of length l from 1 to n is equal to $(n-2)(n-3)\cdots(n-l)$. Thus the total number of paths in K_n from 1 to n is equal to $\sum_{l=1}^{n}(n-2)(n-3)\cdots(n-l)$.

4: (a) Draw a picture of a 3-regular graph which has a bridge.

Solution: Here is one such graph.



(b) Prove that no 4-regular graph has a bridge.

Solution: Let G be a 4-regular graph. Suppose, for a contradiction, that G has a bridge. Let $e = \{u, v\}$ be a bridge. Then the graph $G - e$ has 2 components, one containing u and the other containing v. In the component containing u, all vertices have degree 4 except for u which has degree 3. This is not possible since, as shown in class, every graph has an odd number of vertices of odd degree.

(c) Suppose that G is a graph with exactly two vertices of odd degree, namely a and b. Prove that there exists a path from a to b in G.

Solution: We prove this by induction on q, the number of edges in G. When $q = 1$, in order for a and b to have odd degree we must have $E(G) = \big\{\{a, b\}\big\}$, and so a, b is a path of length 1 from a to b. Let $q \geq 2$ and suppose, inductively, that for every graph H with $q - 1$ edges, if H has exactly two vertices of odd degree then there is a path in H between these two vertices. Let G be any graph with q vertices with exactly two vertices, say a and b, of odd degree. Let e be an edge at a in G, say $e = \{a, u\}$. If $u = b$ then a, u is a path of length 1 from a to b in G. Suppose that $u \neq b$ and note that $\deg(u)$ is even (since only a and b have odd degree). Let H be the graph $G - e$. Then H has $q - 1$ edges and H has exactly two vertices of odd degree, namely u and b (since the degrees of a and u decrease by 1 when we remove the edge e). By the induction hypothesis, there is a path in H from u to b, say $u = v_0, v_1, \cdots, v_l = b$ is such a path. If $a = v_i$ for some i, then $v_i, v_{i+1}, \cdots, v_l = b$ is a path from a to b in H (hence also in G). If $a \neq v_i$ for any i then $a, u = v_0, v_1, \cdots, v_l = b$ is a path from a to b in G.

5: Let G be a graph.

(a) For $a \in V(G)$, let $U(a) = \big\{v \in V(G) \big| a \sim v\big\}$ and let $H(a)$ be the maximal connected subgraph of G containing a. Show that $H(a)$ is the subgraph of G induced by $U(a)$. (The graph $H(a)$ is called *the connected component of G containing a*).

Solution: Write $H = H(a)$ and $U = U(a)$. We must show that $V(H) = U$ and that $E(H) = E(U)$, where $E(U)$ is the set of all edges in G whose endpoints both lie in U. First we show that $V(H) \subseteq U$. Let $v \in V(H)$. Since $a, v \in V(H)$ and H is connected, we have $a \sim v$ in H, hence $a \sim v$ in G, ans so $v \in U(a)$. Thus $V(H) \subseteq U$, as claimed. Next we show that $U \subseteq V(H)$. Let $v \in U$, so we have $a \sim v$ in G. Suppose, for a contradiction that $v \notin V(H)$. Choose a path $a = v_0, v_1, \cdots, v_l = v$ from a to v in G. Since $a \in V(H(a))$ and $v \notin V(H)$ we can choose k so that $v_0, v_1, \cdots, v_{k-1} \in V(H)$ but $v_k \notin V(H(a))$. Let $e = \{v_{k-1}, v_k\}$. Let $H + e$ be the graph with vertex set $V(H + e) = V(H) \cup \{v_k\}$ and edge set $E(H + e) = E(H) \cup \{e\}$. We shall show that $H + e$ is connected, contradicting the maximality of the connected component H. Let $x, y \in V(H + e)$. If x and y both lie in $V(H)$ then $x \sim y$ in H (since H is connected), hence also in $H + e$. Otherwise, one of x and y lies in $V(H)$ and the other is equal to v_k, say $x \in V(H)$ and $y = v_k$. In this case we have $x \sim v_{k-1}$ in H (since $x, v_{k-1} \in V(H)$ and H is connected), hence also in $H + e$, and $v_{k-1} \sim v_k$ in $H + e$ (since $e = \{v_{k-1}, v_k\}$ so v_{k-1}, v_k is a path of length 1 in $H + e$), and so $x \sim y$ in $H + e$. Thus $H + e$ is connected, as claimed, giving the desired contradiction. This shows that $V(H) = U$.

It remains to show that $E(H) = E(U)$. Since all edges of H have both ends in $V(H) = U$, we have $E(H) \subseteq E(U)$. On the other hand, we have $E(U) \subseteq E(H)$ since every edge e of G with both ends in $U = V(H)$ must lie in the edge set $E(H)$ since, if not, then the graph $H+e$ with vertex set $V(H+e) = V(H)$ and edge set $E(H+e) = E(H) \cup \{e\}$ would be a larger connected subgraph of G, contradicting the maximality of H.

(b) Show that for all $a, b \in V(G)$, either $H(a) = H(b)$ or $H(a) \cap H(b) = \emptyset$. (This shows that the connected components of G are disjoint).

Solution: Suppose that $H(a) \cap H(b) \neq \emptyset$. Then $V(H(a)) \cap V(H(b)) \neq \emptyset$, that is $U(a) \cap U(b) \neq \emptyset$. Choose $u \in U(a) \cap U(b)$. Then $a \sim u$ in G and $b \sim u$ in G, so we have $a \sim b$ in G. Thus for any vector $x \in V(G)$, we have $x \in U(a) \iff x \sim a$ in $G \iff x \sim b$ in $G \iff v \in U(b)$, and so $U(a) = U(b)$. By part (a) this implies that $H(a) = H(b)$.