MATH 239 Intro to Combinatorics, Solutions to Assignment 3

1: Given $n, k \in \mathbf{N} = \{0, 1, 2, \cdots\}$, find the number of sequences $(a_1, a_2, \cdots, a_{2k})$ with each $a_i \in \mathbf{N}$ such that $a_1 + a_2 + \cdots + a_k + 2a_{k+1} + 2a_{k+2} + \cdots + 2a_{2k} = n$. In particular, what is the number of such sequences when $k = 5$ and $n = 7$?

Solution: Let S be the set of sequences $(b_1, \cdots, b_k, b_{k+1}, \cdots, b_{2k})$ with $b_i \in \mathbf{N}$ for $1 \le i \le k$ and $b_i \in 2\mathbf{N} = \{0, 2, 4, \cdots\}$ with weight $w(b_1, \cdots, b_{2k}) = b_1 + \cdots + b_{2k}$. Notice that the required number of sequences $(a_1, \cdots, a_k, a_{k+1}, \cdots, a_{2k})$ is equal to $|S_n|$, as we can see using the bijection given by $b_i = a_i$ for $1 \le i \le k$ and $b_i = 2a_i$ for $k < i \le 2k$. We have

$$\begin{aligned}
\phi_{\mathbf{N}} &= \frac{1}{1-x}\ ,\ \phi_{2\mathbf{N}} = \frac{1}{1-x^2} \\
\phi_S &= \left(\phi_{\mathbf{N}}\right)^k \left(\phi_{2\mathbf{N}}\right)^k = \left(\frac{1}{1-x}\right)^k \left(\frac{1}{1-x^2}\right)^k = (1-x)^{-k}(1-x^2)^{-k} \\
&= \sum_{i \ge 0} \binom{k+i-1}{k-1} x^i \sum_{j \ge 0} \binom{k+j-1}{k-1} x^{2j} \\
&= \sum_{i,j \ge 0} \binom{k+i-1}{k-1} \binom{k+j-1}{k-1} x^{i+2j}
\end{aligned}$$

and so

$$|S_n| = [x^n]\phi = \sum_{j \ge 0} \binom{k+n-2j-1}{k-1} \binom{k+j-1}{k-1} .$$

In particular, when $k = 5$ and $n = 7$ we have

$$\begin{aligned}
|S_7| &= \sum_{j \ge 0} \binom{11-2j}{4} \binom{4+j}{4} \\
&= \tbinom{11}{4}\tbinom{4}{4} + \tbinom{9}{4}\tbinom{5}{4} + \tbinom{7}{4}\tbinom{6}{4} + \tbinom{5}{4}\tbinom{7}{4} \\
&= 330 \cdot 1 + 126 \cdot 5 + 35 \cdot 15 + 5 \cdot 35 = 1660 .
\end{aligned}$$

2: For a positive integer n, let c_n be the number of compositions of n into an even number of parts each of which is odd. Find the generating function $\sum c_n x^n$ expressed as a rational function, obtain a recursion formula for c_n, and use the recursion formula to find c_{10}.

Solution: Let S_k be the set of compositions of positive integers into k odd parts, that is S_k is the set of integer sequences $(a_1, \cdots, a_k)$ with each $a_i \in \{1,3,5,7,\cdots\}$, with weight given by $w(a_1,\cdots,a_k) = a_1 + \cdots + a_k$. Let S be the set of compositions of positive integers into an even number of odd parts, that is $S = S_2 \cup S_4 \cup S_6 \cup \cdots$. We have

$$\begin{aligned}
\phi_{\{1,3,5,\cdots\}} &= x + x^3 + x^5 + \cdots = \frac{x}{1-x^2} \\
\phi_{S_k} &= \left(\frac{x}{1-x^2}\right)^k = x^k(1-x^2)^{-k} \\
\phi_S &= \left(\frac{x}{1-x^2}\right)^2 + \left(\frac{x}{1-x^2}\right)^4 + \left(\frac{x}{1-x^2}\right)^6 + \cdots \\
&= \frac{\left(\frac{x}{1-x^2}\right)^2}{1-\left(\frac{x}{1-x^2}\right)^2} = \frac{x^2}{(1-x^2)^2 - x^2} = \frac{x^2}{1-3x^2+x^4}\,.
\end{aligned}$$

If we write $\phi_S(x) = \sum_{i\geq 0} c_n x^n$ then we have $c_i = 0$ for all odd indices i, and

$$\begin{gathered}
(1-3x^2+x^4)(c_0 + c_2x^2 + c_4x^4 + \cdots) = x^2 \\
c_0 + (c_2 - 3c_0)x^2 + (c_4 - 3c_2 + c_0)x^4 + \cdots + (c_n - 3c_{n-2} + c_{n-4})x^n + \cdots = x^2
\end{gathered}$$

and so $c_0 = 0$, $c_2 = 0$ and for n even with $n \geq 4$ we have $c_n = 3c_{n-2} - c_{n-4}$. In particular we have

n	0	2	4	6	8	10
c_n	0	1	3	8	21	55

3: Given positive integers n and k, find the number of integer sequences $(a_1, a_2, \cdots, a_k)$ with $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ such that $a_i = i \pmod 3$ for all i. Express your answer in simplified form using the floor function.

Solution: Using the bijection $\psi(a_1, a_2, \cdots, a_k) = (a_1, a_2 - a_1, a_3 - a_2, \cdots, a_k - a_{k-1}, n - a_k) = (b_1, \cdots, b_k, b_{k+1})$, we see that the number of such sequences $(a_1, \cdots, a_k)$ is equal to the number of sequences $(b_1, b_2, \cdots, b_k, b_{k+1})$ with each $b_i \geq 1$ with $b_i = 1 \pmod 3$, that is $b_i \in \{1, 4, 7, 10, \cdots\}$, for $1 \leq i \leq k$, with $b_{k+1} \geq 0$ and with $b_1 + b_2 + \cdots b_{k+1} = n$. We let S be the set of sequences $(b_1, \cdots, b_k, b_{k+1})$ with $b_i \in \{1, 4, 7, 10, \cdots\}$ for $1 \leq i \leq k$ and with $b_{k+1} \geq 0$, where the weight is given by $w(b_1, \cdots, b_k, b_{k+1}) = b_1 + \cdots, b_k + b_{k+1}$. We need to find $|S_n|$. We have

$$\begin{aligned}
\phi_{\{1,4,7,10,\cdots\}} &= x + x^4 + x^7 + x^{10} + \cdots = \frac{x}{1 - x^3} \\
\phi_{\{0,1,2,3,\cdots\}} &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x} \\
\phi_S &= \left(\frac{x}{1 - x^3}\right)^k \left(\frac{1}{1 - x}\right) = \left(\frac{x}{1 - x^3}\right)^k \left(\frac{1 + x + x^2}{1 - x^3}\right) = x^k (1 + x + x^2)(1 - x^3)^{-k} \\
&= x^k (1 + x + x^2) \sum_{i \geq 0} \binom{k + i}{k} x^{3i} .
\end{aligned}$$

To find $|S_n| = [x^n]\phi$, we choose i so that $n = k + 3i$, $n = k + 1 + 3i$ or $n = k + 2 + 3i$, that is we choose $i = \frac{n-k}{3}$, $i = \frac{n-k-1}{3}$ or $i = \frac{n-k-2}{3}$, depending on whether $n - k = 0$, 2 or 1 modulo 3. Note that in all cases, we have $i = \lfloor \frac{n-k}{3} \rfloor$. Thus we have

$$|S_n| = [x^n]\phi = \binom{k + \lfloor \frac{n-k}{3} \rfloor}{k} .$$

4: For each $n \in \mathbf{N}$, let c_n be the number of binary strings of length n which do not contain either 0000 or 1111 as substrings. Find the generating function $\sum c_n x^n$ expressed as a rational function, obtain a recursion formula for c_n, and find c_6.

Solution: Let S be the set of all binary strings which do not contain either 000 or 111 as substrings. Note that

$$S = \{\epsilon, 1, 11, 111\}\big(\{0, 00, 000\}\{1, 11, 111\}\big)^*\{\epsilon, 0, 00, 000\}\,.$$

We have

$$\begin{aligned}
\phi_{\{\epsilon,1,11,111\}} &= \phi_{\{\epsilon,0,00,000\}} = 1 + x + x^2 + x^3 \\
\phi_{\{0,00,000\}} &= \phi_{\{1,11,111\}} = x + x^2 + x^3 \\
\phi_S &= (1 + x + x^2 + x^3)^2\,\frac{1}{1 - (x + x^2 + x^3)^2} \\
&= \frac{(1 + x + x^2 + x^3)^2}{\big(1 - (x + x^2 + x^3)\big)\big(1 + (x + x^2 + x^3)\big)} \\
&= \frac{1 + x + x^2 + x^3}{1 - x - x^2 - x^3}\,.
\end{aligned}$$

If we write $\phi_S = \sum c_n x^n$ then we have

$$(1 - x - x^2 - x^3)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) = 1 + x + x^2 + x^3$$

$$c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2 + (c_3 - c_2 - c_1 - c_0)x^3 + \cdots + (c_n - c_{n-1} + c_{n-2} + c_{n-3})x^n + \cdots = 1 + x + x^2 + x^3$$

and so $c_0 = 1$, $c_1 = 2$, $c_2 = 4$, $c_3 = 8$ and for $n \geq 4$ we have $c_n = c_{n-1} + c_{n-2} + c_{n-3}$. In particular, the first few values of c_n are as follows

n	0	1	2	3	4	5	6
c_n	1	2	4	8	14	26	48

5: Find the generating function with respect to length, expressed as a rational function, for the set of binary strings in which no 0-block is followed by a 1-block of greater length. In particular, find the number of such sequences of length 8.

Solution: Let S be the set of binary strings in which no 0-block is followed by a 1-block of greater length. Let $M = \{01, 0011, 000111, 00001111, \cdots\}$ and note that

$$S = \{\epsilon, 1, 11, 111, \cdots\}\big(\{\epsilon, 0, 00, 000, \cdots\}M\big)^*\{\epsilon, 0, 00, 000, \cdots\}\,.$$

We have

$$\begin{aligned}\phi_{\{\epsilon,1,11,111,\cdots\}} = \phi_{\{\epsilon,0,00,000,\cdots\}} &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\ \phi_M &= x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2} \\ \phi_S &= \left(\frac{1}{1-x}\right)^2 \frac{1}{1 - \left(\frac{1}{1-x}\right)\left(\frac{x^2}{1-x^2}\right)} = \frac{1}{(1-2x+x^2) - \frac{x^2}{1+x}} \\ &= \frac{1+x}{1-x-2x^2+x^3}\,.\end{aligned}$$

Write $\phi_S = \sum c_n x^n$. Then we have

$$(1-x-2x^2+x^3)(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots) = 1 + x$$

$$c_0 + (c_1 - c_0)x + (c_2 - c_1 - 2c_0)x^2 + (c_3 - c_2 - 2c_1 + c_0)x^3 + \cdots + (c_n - c_{n-1} - 2c_{n-2} + c_{n-3})x^n + \cdots = 1 + x$$

and so $c_0 = 1$, $c_1 = 2$, $c_2 = 4$ and $c_n = c_{n-1} + 2c_{n-2} - c_{n-3}$ for $n \geq 3$. In particular, the first few values of c_n are

n	0	1	2	3	4	5	6	7	8
c_n	1	2	4	7	13	23	42	75	136