MATH 239 Intro to Combinatorics, Solutions to Assignment 2

1: (a) By describing a method of counting the number of k-element subsets of $\{1, 2, \cdots, n\}$, give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} .$$

Solution: Let S be the set of all k-element subsets of $\{1, 2, \cdots, n\}$, let $U = \big\{A \in S \big| n \in A\big\}$ and let $V = \big\{A \in S \big| n \notin A\big\}$. Note that $|S| = |U| + |V|$. We have a bijection from U to the set of all $(k-1)$-element subsets of $\{1, 2, \cdots, n-1\}$ given by $\phi(A) = A \setminus \{n\}$ and so $|U| = \binom{n-1}{k-1}$. Also note that V is equal to the set of all k-element subsets of $\{1, 2, \cdots, n-1\}$ and so $|V| = \binom{n-1}{k}$. Thus $|S| = |U| + |V| = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(b) By counting the number of ways to choose sets A and B with $A \subseteq B \subseteq \{1, 2, \cdots, n\}$ and $|A| = m$ in two different ways, show that

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m} .$$

Solution: For each fixed k with $m \leq k \leq n$, the number of ways of choosing a k-element subset B of $\{1, 2, \cdots, n\}$ is equal to $\binom{n}{k}$, and having chosen B, the number of ways of choosing an m-element subset $A \subseteq B$ is equal to $\binom{k}{m}$. Thus the total number of ways to choose A and B is equal to

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} .$$

On the other hand, the number of ways to choose an m-element subset $A \subset \{1, 2, \cdots, n\}$ is equal to $\binom{n}{m}$, and having chosen A, the number of ways to choose B with $A \subseteq B \subseteq \{1, 2, \cdots, n\}$ is equal to 2^{n-m} (since $B \setminus A$ can be any of the 2^{n-m} subsets of $\{1, 2, \cdots, n\} \setminus A$). Thus the total number of ways to choose A and B is also equal to

$$2^{n-m}\binom{n}{m} .$$

2: (a) Evaluate $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}$.

Solution: By the Binomial Theorem, we have $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}=\left(1+\tfrac{1}{2}\right)^n=\left(\tfrac{3}{2}\right)^n$.

(b) Evaluate $\displaystyle\sum_{k=0}^{n}\binom{2n}{2k}\frac{1}{2^k}$.

Solution: By the Binomial Theorem, we have

$$\binom{2n}{0}+\binom{2n}{1}\tfrac{1}{\sqrt{2}}+\binom{2n}{2}\tfrac{1}{\sqrt{2}^2}+\binom{2n}{3}\tfrac{1}{\sqrt{2}^3}+\binom{2n}{4}\tfrac{1}{\sqrt{2}^4}+\cdots+\binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}}=\left(1+\tfrac{1}{\sqrt{2}}\right)^{2n}$$

and

$$\binom{2n}{0}-\binom{2n}{1}\tfrac{1}{\sqrt{2}}+\binom{2n}{2}\tfrac{1}{\sqrt{2}^2}-\binom{2n}{3}\tfrac{1}{\sqrt{2}^3}+\binom{2n}{4}\tfrac{1}{\sqrt{2}^4}-\cdots+\binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}}=\left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}.$$

Adding these gives

$$2\left(\binom{2n}{0}+\binom{2n}{2}\tfrac{1}{\sqrt{2}^2}+\binom{2n}{4}\tfrac{1}{\sqrt{2}^4}+\cdots+\binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}}\right)=\left(1+\tfrac{1}{\sqrt{2}}\right)^{2n}+\left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}.$$

Thus $\displaystyle\sum_{k=0}^{n}\binom{2n}{2k}\tfrac{1}{2^k}=\tfrac{1}{2}\left(\left(1+\tfrac{1}{\sqrt{2}}\right)^{2n}+\left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}\right)$.

3: (a) Let $\phi(x) = \dfrac{(1+3x)^{4/3}}{1+x}$. Find $\big[x^k\big]\phi(x)$ for $k = 0, 1, \cdots, 5$.

Solution: Note that this problem is equivalent to finding the 5^{th} Taylor polynomial $T_5(x)$ for $\phi(x)$, centered at 0. We have

$$\begin{aligned}(1+3x)^{4/3} &= 1 + \tfrac{4}{3}(3x) + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)}{2!}(3x)^2 + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{3!}(3x)^3 + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{4!}(3x)^4 \\ &\quad + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{5!}(3x)^5 + \cdots \\ &= 1 + 4x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{5}{3}x^4 - \tfrac{8}{3}x^5 + \cdots ,\end{aligned}$$

and so

$$\begin{aligned}\phi(x) &= \frac{1}{1+x}\,(1+3x)^{4/3} \\ &= \Big(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\Big)\Big(1 + 4x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{5}{3}x^4 - \tfrac{8}{3}x^5 + \cdots\Big) \\ &= 1 + \big(4-1\big)x + \big(2-4+1\big)x^2 + \big(-\tfrac{4}{3} - 2 + 4 - 1\big)x^3 + \big(\tfrac{5}{3} + \tfrac{4}{3} + 2 - 4 + 1\big)x^4 \\ &\qquad + \big(-\tfrac{8}{3} - \tfrac{5}{3} - \tfrac{4}{3} - 2 + 4 - 1\big)x^5 + \cdots \\ &= 1 + 3x - x^2 - \tfrac{1}{3}x^3 + 2x^4 - \tfrac{14}{3}x^5 + \cdots .\end{aligned}$$

Thus the 5^{th} Taylor polynomial is $T_5(x) = 1 + 3x - x^2 - \frac{1}{3}x^3 + 2x^4 - \frac{14}{3}x^5$.

(b) Find a closed form formula for the function $\phi(x) = \sum\limits_{n=0}^{\infty} \frac{3+(-1)^n}{2}\, x^n$.

Solution: Notice that this is the sum of two geometric series. We have

$$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2}\, x^n = \sum_{n=0}^{\infty} \tfrac{3}{2}\, x^n + \sum_{n=0}^{\infty} \tfrac{1}{2}\,(-x)^n = \frac{\frac{3}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} = \frac{\frac{1}{2}\big(3(1+x) + (1-x)\big)}{1-x^2} = \frac{2+x}{1-x^2}\,.$$

4: (a) Let S be the set of binary sequences of length 5, where the weight $w(e_1, e_2, \cdots, e_5)$ is the number of occurrences of the substring 01 in the sequence $(e_1, e_2, \cdots, e_5)$. Find the generating series $\phi_S(x)$.

Solution: We list all 32 binary sequences $a = a_1a_2a_3a_4a_5$, and for each sequence a, we count the number of occurrences of the substring 01.

a	$w(a)$	a	$w(a)$
00000	0	10000	1
00001	1	10001	1
00010	1	10010	1
00011	1	10011	0
00100	1	10100	1
00101	2	10101	1
00110	1	10110	2
00111	1	10111	1
01000	1	11000	1
01001	2	11001	1
01010	2	11010	1
01011	2	11011	0
01100	1	11100	0
01101	2	11101	0
01110	1	11110	1
01111	1	11111	0

We see that there are 6 sequences a with $w(a) = 0$, there are 20 sequences a with $w(a) = 20$ and there are 6 sequences a with $w(a) = 2$, and so the generating function for S is $\phi_S(x) = 6 + 20x + 6x^2$.

(b) Let S be the set of all subsets of $\{1, 2, 3, 4, 5\}$, where the weight $w(A)$ is the number of consecutive integers in the subset A. Find the generating series $\phi_S(x)$.

Solution: We list all 32 subsets of $\{1, 2, 3, 4, 5\}$, and for each subset A we count the number of consecutive elements in A. In the list below, we omit brackets and commas, writing the set $\{a_1, a_2, \cdots, a_k\}$ as $a_1a_2 \cdots a_k$.

A	$w(A)$	A	$w(A)$
$\emptyset$	0	123	2
1	0	124	1
2	0	125	1
3	0	134	1
4	0	135	0
5	0	145	1
12	1	234	2
13	0	235	1
14	0	245	1
15	0	345	2
23	1	1234	3
24	0	1235	2
25	0	1245	2
34	1	1345	2
35	0	2345	3
45	1	12345	4

We see that there are 13 sets A with $w(A) = 0$, 10 with $w(A) = 1$, 6 with $w(A) = 2$, 2 with $w(A) = 3$ and 1 with $w(A) = 4$, and so the generating series is $\phi_S(x) = 13 + 10x + 6x^2 + 2x^3 + x^4$.

5: (a) Use generating series to find the number of sequences (a_1, a_2, a_3, a_4), with $a_i \in \{0, 1, 3\}$ for all i, such that $a_1 + a_2 + a_3 + a_4 = n$.

Solution: Let S be the set of sequences (a_1, a_2, a_3, a_4) with each $a_i \in \{0, 1, 3\}$, with weight function $w(a_1, \cdots, a_4) = a_1 + \cdots + a_4$. Then we have

$$\begin{aligned}
\phi_{\{0,1,3\}}(x) &= 1 + x + x^3 \\
\phi_S(x) &= (1 + x + x^3)^4 = \big((1 + x + 0x^2 + x^3)^2\big)^2 \\
&= \big(1 + 2x + 1x^2 + 2x^3 + 2x^4 + 0x^5 + 1x^6\big)^2 \\
&= 1 + 4x + 6x^2 + 8x^3 + 13x^4 + 12x^5 + 10x^6 + 12x^7 + 6x^8 + 4x^9 + 4x^{10} + 0x^{11} + 1x^{12}\,.
\end{aligned}$$

The n^{th} coefficient of $\phi_S(x)$ is the required number of sequences.

(b) Use generating series to find the number of ways to select n letters from $\{A, B, C, D\}$, with repetition allowed and order unimportant, such that A and B can each be chosen at most once and C and D can be chosen any number of times.

Solution: Let a_1, a_2, a_3 and a_4 be the number of times we select the letters A, B, C and D respectively. Let S be the set of 4-tuples (a_1, a_2, a_3, a_4) with $a_1, a_2 \in \{0, 1\}$ and $\{a_3, a_4 \in \mathbf{N} = \{0, 1, 2, \cdots\}$, with weight function $w(a_1, \cdots, a_4) = a_1 + \cdots + a_4$. We must find $|S_n|$. We have $\phi_{\{0,1\}}(x) = 1 + x$ and $\phi_{\mathbf{N}}(x) = 1 + x + x^2 + \cdots$ and so

$$\begin{aligned}
\phi_S(x) &= (1 + x)^2\big(1 + x + x^2 + x^3 + \cdots\big)^2 \\
&= (1 + 2x + x^2)\big(1 + 2x + 3x^2 + 4x^3 + \cdots\big) \\
&= 1 + 4x + (3 + 2\cdot 2 + 1)x^2 + (4 + 2\cdot 3 + 2)x^3 + \cdots + \big((n + 1) + 2\cdot n + (n - 1)\big)x^n + \cdots \\
&= 1 + 4x + 8x^2 + 12x^3 + \cdots + (4n)x^n + \cdots\,.
\end{aligned}$$

Thus $|S_n| = [x^n]\phi_S(x) = 4n$.