MATH 239 Intro to Combinatorics, Solutions to Assignment 1

1: Given a positive integer n, find the number of sequences $(a_1, a_2, \cdots, a_n)$, with $a_i \in \{1,2,3\}$ for all i, which do not contain 11, 22 or 33 as substrings.

Solution: There are 3 choices for a_1 (since $a_1 \in \{1,2,3\}$) and there are 2 choices for each of $a_2, a_3, \cdots a_n$ (since a_k cannot be equal to a_{k-1} so $a_k \in \{1,2,3\} \setminus \{a_{k-1}\}$), and so altogether there are $3 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 3 \cdot 2^{n-1}$ such sequences $(a_1, a_2, \cdots, a_n)$.

2: Given a positive integer n, find the number of binary sequences of length n which do not contain either 000 or 111 as a subsequence.

Solution: Let c_n be the number of such sequences. Such a sequence must end with 00, 01, 10 or 11. By symmetry, the number of sequences that end with 00 is the same as the number that end with 11 and the number that end with 01 is the same as the number that end with 10. Let a_n be the number of such sequences that end with 00 and let b_n be the number that end with 01. Note that $c_n = 2a_n + 2b_n$. Every such sequence which ends with 00 must end with 100 (since it cannot end with 000) and so we have $a_n = b_{n-1}$. On the other hand, a sequence which ends with 01 can end either 001 or with 101, and so we have $b_n = a_{n-1} + b_{n-1} = b_{n-2} + b_{n-1}$. Since $b_2 = 1$ and $b_3 = 2$ and $b_n = b_{n-1} + b_{n-2}$ (the same recurrence as the Fibonacci sequence), we have $b_n = F_n$, the n^{th} Fibonacci number. Thus we have $c_n = 2a_n + 2b_n = 2b_{n-1} + 2b_n = 2(F_{n-1} + F_n) = 2F_{n+1}$.

3: Given positive integers n and k, determine the number of sequences $(a_1, a_2, \cdots, a_k)$, with $-1 \leq a_i \in \mathbf{Z}$ for all i, such that $a_1 + a_2 + \cdots + a_k = n$.

Solution: Let A_k be the set of such sequences $(a_1, a_2, \cdots, a_k)$ and let B_k be the set of compositions of $n+2k$ into k parts, that is the set of sequences $(b_1, b_2, \cdots, b_k)$ with $1 \leq b_i$ for all i and $b_1 + b_2 + \cdots + b_k = n + 2k$. We have a bijection $\phi : A_k \to B_k$ given by $\phi(a_1, a_2, \cdots, a_k) = (a_1 + 2, a_2 + 2, \cdots, a_k + 2)$, and so we have $|A_k| = |B_k| = \binom{n+2k-1}{k-1}$.

4: Given a positive integer $n \geq 2$, show that the total number of sequences $(a_1, a_2, \cdots, a_k)$ with $1 \leq k$, $2 \leq a_i \in \mathbf{Z}$ for all i, and $a_1 + a_2 + \cdots + a_k = n$ is equal to F_{n-1}, the $(n-1)^{\text{st}}$ Fibonacci number.

Solution: Let A_k be the set of such sequences $(a_1, a_2, \cdots, a_k)$ of length k, and let B_k be the set of compositions of $n-k$ into k parts. We have a bijection $\phi : A_k \to B_k$ given by $\phi(a_1, a_2, \cdots, a_k) = (a_1 - 1, a_2 - 1, \cdots, a_n - 1)$ and so $|A_k| = |B_k| = \binom{n-k-1}{k-1}$. Let $c_n = \sum_{k \geq 1} |A_k| = \binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \cdots$. We must show that $c_n = F_{n-2}$. We have $c_2 = \binom{0}{0} = 1 = F_1$ and $c_3 = \binom{1}{0} = 1 = F_2$ and so it suffice to show that $c_n = c_{n-1} + c_{n-2}$. And indeed, using the formula $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, we have

$$\begin{aligned}
c_n &= \binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{4} + \cdots \\
&= 1 + \left(\binom{n-4}{0} + \binom{n-4}{1}\right) + \left(\binom{n-5}{1} + \binom{n-5}{2}\right) + \left(\binom{n-6}{2} + \binom{n-6}{3}\right) + \cdots \\
&= \left(1 + \binom{n-4}{1} + \binom{n-5}{2} + \binom{n-6}{3} + \cdots\right) + \left(\binom{n-4}{0} + \binom{n-5}{1} + \binom{n-6}{2} + \cdots\right) \\
&= c_{n-1} + c_{n-2}\,.
\end{aligned}$$

5: Let c_n be the number of sequences $(a_1, a_2, \cdots, a_n)$, with each $a_i \in \{1, 2\}$, such that $a_1 + a_2 + \cdots + a_k = n$ for some $k \leq n$.

(a) Show that $c_n = c_{n-1} + 2c_{n-2}$.

Solution: Let C_n be the set of all such sequences. Let A_n^1 be the set of all such sequences $(a_1, a_2, \cdots, a_n)$ where $a_1 + a_2 + \cdots + a_k = n$ for some k with $a_k = 1$, let $A_n^{2,1}$ be the set of all such sequences where $a_1 + a_2 + \cdots + a_k = n$ for some k with $a_k = 2$ and $a_n = 1$, and let $A_n^{2,2}$ be the set of such sequences where $a_1 + a_2 + \cdots + a_k = n$ for some k with $a_k = 2$ and $a_n = 2$. We have a bijection $\phi : A_n^1 \to C_{n-1}$ and we have bijections $\psi : A_n^{2,1} \to C_{n-2}$ and $\psi : A_n^{2,2} \to C_{n-2}$ given by

$$\phi(a_1, a_2, \cdots, a_{k-1}, a_k = 1, a_{k+1}, \cdots, a_n) = (a_1 \cdots, a_{k-1}, a_{k-2}, \cdots, a_n) \text{ and}$$
$$\psi(a_1, \cdots, a_{k-1}, a_k = 2, a_{k+1}, \cdots, a_{n-1}, a_n) = (a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{n-1}) .$$

Thus $c_n = |C_n| = |A_n^1 \cup A_n^{2,1} \cup A_n^{2,2}| = |A_n^1| + |A_n^{2,1}| + |A_n^{2,2}| = |C_{n-1}| + 2|C_{n-2}| = c_{n-1} + 2c_{n-2}$.

(b) Find a formula for c_n in terms of n.

Solution: The characteristic polynomial is $f(x) = x^2 - x - 2 = (x-2)(x+1)$. It has roots 2 and -1, so we must have $c_n = A \cdot 2^n + B \cdot (-1)^n$ for some constants A and B. Since $c_1 = 1$ and $c_2 = 3$ (indeed $C_1 = \big\{(1)\big\}$ and $C_2 = \big\{(1,1), (2,1), (2,2)\big\}$), putting $n = 1$ and $n = 2$ into the equation $c_n = A \cdot 2^n + B \cdot (-1)^n$ gives $2A - B = 1$ and $4A + B = 3$. Solving these two equations gives $A = \frac{2}{3}$ and $B = \frac{1}{3}$ and so we have

$$c_n = \tfrac{2}{3} \cdot 2^n + \tfrac{1}{3} \cdot (-1)^n = \frac{2^{n+1} + (-1)^n}{3} .$$