MATH 239 Intro to Combinatorics, Assignment 4 Due Thurs Feb 22

1: (a) Find the incidence matrix for the graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set $E = \big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 6\}, \{3, 5\}, \{4, 5\}, \{4, 6\}\big\}$.

(b) Using the 6 points in the plane given by $p_k = e^{i\,k\pi/3}$ for $k = 1, 2, \cdots, 6$ (these are the vertices of a regular hexagon), draw a picture of the graph with adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

(c) Let A be the adjacency matrix for a graph G. Show that $(A^n)_{k,l}$ is equal to the number of walks of length n from v_k to v_l in G.

2: (a) Up to isomorphism, there are 20 graphs with 5 vertices with at most 5 edges. Without proof, draw a picture of one graph for each of these 20 isomorphism classes.

(b) Find two connected graphs with 5 vertices which are not isomorphic but have the same number of vertices of each degree.

3: (a) Let $S_{n,k,i}$ be the graph whose vertex set V is the set of k-element subsets of $\{1, 2, \cdots, n\}$ and whose edge set E is the set of 2-element sets $\{A, B\}$ with $A, B \in V$ such that $|A \cap B| = i$. Find the number of edges in $S_{n,k,i}$ and find the value of r such that $S_{n,k,i}$ is r-regular.

(b) Let K_n be the complete graph with vertex set $V = \{1, 2, \cdots, n\}$ whose edge set E is the set of all 2-element sets $\{a, b\}$ with $a, b \in V$. Find the number of subgraphs of K_n and find the number of paths from 1 to n in K_n.

4: (a) Draw a picture of a 3-regular graph which has a bridge.

(b) Prove that no 4-regular graph has a bridge.

(c) Suppose that G is a graph with exactly two vertices of odd degree, namely a and b. Prove that there exists a path from a to b in G.

5: Let G be a graph.

(a) For $a \in V(G)$, let $U(a) = \big\{v \in V(G) \big| a \sim v\big\}$ and let $H(a)$ be the maximal connected subgraph of G containing a. Show that $H(a)$ is the subgraph of G induced by $U(a)$. (The graph $H(a)$ is called *the connected component of* G *containing* a).

(b) Show that for all $a, b \in V(G)$, either $H(a) = H(b)$ or $H(a) \cap H(b) = \emptyset$. (This shows that the connected components of G are disjoint).