MATH 239 Intro to Combinatorics, Assignment 2 Due Thurs Feb 9

1: (a) By describing a method of counting the number of k-element subsets of $\{1, 2, \cdots, n\}$, give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(b) By counting the number of ways to choose sets A and B with $A \subseteq B \subseteq \{1, 2, \cdots, n\}$ and $|A| = m$ in two different ways, show that

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}.$$

2: (a) Evaluate $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k}$.

(b) Evaluate $\displaystyle\sum_{k=0}^{n} \binom{2n}{2k} \frac{1}{2^k}$.

3: (a) Let $\phi(x) = \dfrac{(1+3x)^{4/3}}{1+x}$. Find $\left[x^k\right]\phi(x)$ for $k = 0, 1, \cdots, 5$.

(b) Find a closed form formula for the function $\phi(x) = \sum_{n=0}^{\infty} \frac{3+(-1)^n}{2}\, x^n$.

4: (a) Let S be the set of binary sequences of length 5, where the weight $w(e_1, e_2, \cdots, e_5)$ is the number of occurrences of the substring 01 in the sequence $(e_1, e_2, \cdots, e_5)$. Find the generating series $\phi_S(x)$.

(b) Let S be the set of all subsets of $\{1, 2, 3, 4, 5\}$, where the weight $w(A)$ is the number of consecutive integers in the subset A. Find the generating series $\phi_S(x)$.

5: (a) Use generating series to find the number of sequences (a_1, a_2, a_3, a_4), with $a_i \in \{0, 1, 3\}$ for all i, such that $a_1 + a_2 + a_3 + a_4 = n$.

(b) Use generating series to find the number of ways to select n letters from $\{A, B, C, D\}$, with repetition allowed and order unimportant, such that A and B can each be chosen at most once and C and D can be chosen any number of times.