MATH 218 Differential Equations for Engineers, Solutions to the Midterm Test, Fall 2008

[5] **1:** (a) Solve the IVP $x\,y' + 2y = 6$ with $y(1) = 2$.

Solution: This DE is linear since we can write it as $y' + \frac{2}{x}\,y = \frac{6}{x}$. An integrating factor is given by $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$, and the solution to the DE is $y = \frac{1}{x^2}\int 6x\,dx = \frac{1}{x^2}(3x^2 + c) = 3 + \frac{c}{x^2}$. Put in $x = 1$ and $y = 2$ to get $2 = 3 + c$ so $c = -1$ and the solution to the IVP is $y = 3 - \frac{1}{x^2}$.

(b) Solve the IVP $y' = \frac{x}{y} + \frac{y}{x}$ with $y(1) = 2$.

Solution: This DE is homogeneous. We make the substitution $u = \frac{y}{x}$, $y = xu$, $y' = u + xu'$. The DE becomes $u + xu' = \frac{1}{u} + u$, that is $xu' = \frac{1}{u}$. This new DE is separable, since we can write it as $u\,u' = \frac{1}{x}$. Integrate both sides (with respect to x) to get $\int u\,du = \int \frac{1}{x}\,dx \Longrightarrow \frac{1}{2}\,u^2 = \ln x + c \Longrightarrow u^2 = 2\ln x + 2c$. Put in $x = 1$, $y = 2$, $u = \frac{y}{x} = 2$ to get $4 = 2c$ so $u^2 = 2\ln x + 4$. Thus $\frac{y}{x} = u = \sqrt{2\ln x + 4}$, so $y = x\sqrt{2\ln x + 4}$.

[5] **2:** In a chemical reaction, 1 g of substance A reacts with 1 g of substance B to produce 2 g of substance C. Suppose that 2 g of substance A and 3 g of substance B are combined at time $t = 0$ seconds. Let $a(t)$, $b(t)$ and $c(t)$ be the amounts, in grams, of the three substances at time t, and suppose that $c(t)$ satisfies the DE $c'(t) = 4a(t)b(t)$. Find a formula for $c(t)$.

Solution: To produce c grams of C we must use up $\frac{1}{2}\,c$ grams of A and $\frac{1}{2}c$ grams of B, and so we have $a(t) = 2 - \frac{1}{2}\,c(t)$ and $b(t) = 3 - \frac{1}{2}\,c(t)$. Thus $c(t)$ satisfies the DE $c' = 4ab = 4\left(2 - \frac{1}{2}c\right)\left(3 - \frac{1}{2}b\right) = (4-c)(6-c)$. This DE is separable as we can write it as $\frac{c'}{(4-c)(6-c)} = 1$. Integrate both sides (with respect to t) to get $\int \frac{dc}{(4-c)(6-c)} = \int 1\,dt \Longrightarrow \int \frac{\frac{1}{2}}{4-c} - \frac{\frac{1}{2}}{6-c}\,dc = \int 1\,dc \Longrightarrow \frac{1}{2}\ln\frac{6-c}{4-c} = t + b$. Put in $t = 0$, $c = 0$ to get $\frac{1}{2}\ln\frac{3}{2} = b$, so we have

$$\frac{1}{2}\ln\frac{6-c}{4-c} = t + \frac{1}{2}\ln\frac{3}{2} \Longrightarrow \ln\frac{6-c}{4-c} = 2t + \ln\frac{3}{2} \Longrightarrow \frac{6-c}{4-c} = \frac{3}{2}\,e^{2t} \Longrightarrow 12 - 2c = 12e^{2t} - 3ce^{2t}$$
$$\Longrightarrow c(3e^{2t} - 2) = 12(e^{2t} - 1) \Longrightarrow c = \frac{12(e^{2t} - 1)}{3e^{2t} - 2}\,.$$

[5] **3:** Solve the IVP $x^2y'' = \left(y'\right)^2$ with $y(1) = 3$ and $y'(1) = 2$.

Solution: Notice that the dependent variable y does not occur in the DE, so we make the substitution $y' = u$, $y'' = u'$. The DE becomes $x^2u' = (u)^2$. This DE is separable since we can write it as $\frac{u'}{u} = \frac{1}{x^2}$. Integrate both sides (with respect to x) to get $\int \frac{du}{u^2} = \int \frac{dx}{x^2} \Longrightarrow -\frac{1}{u} = -\frac{1}{x} + b$. Put in $x = 1$, $y = 3$ and $u = y' = 2$ to get $-\frac{1}{2} = -1 + b$ so $b = \frac{1}{2}$, and we have $-\frac{1}{u} = -\frac{1}{x} + \frac{1}{2} = \frac{x-2}{2x}$, that is $y' = u = \frac{2x}{2-x}$. Thus $y = \int \frac{2x\,dx}{2-x} = \int -2 + \frac{4}{2-x}\,dx = -2x - 4\ln(2-x) + c$. Put in $x = 1$ and $y = 3$ to get $3 = -2 + c$ so $c = 5$ and the solution is $y = 5 - 2x - 4\ln(2-x)$.

[5] **4:** Find the general solution to the DE $x^2y'' + 3xy' + y = 0$ for $x > 0$, given that $y_1(x) = \frac{1}{x}$ is one solution to the DE.

Solution: We try $y = y_2 = \frac{1}{x}\,u$. Then $y' = -\frac{1}{x^2}\,u + \frac{1}{x}\,u'$ and $y'' = \frac{2}{x^3}\,u - \frac{2}{x^2}\,u' + \frac{1}{x}\,u''$. Put these in the DE to get

$$\begin{aligned} 0 &= x^2y'' + 3xy' + y \\ &= \tfrac{2}{x}u - 2u' + xu'' - \tfrac{3}{u}u + 3u' + \tfrac{1}{x}u \\ &= xu'' + u'\,. \end{aligned}$$

Let $u' = v$, $u'' = v'$. The DE becomes $v' + \frac{1}{x}v = 0$. This is linear (its also separable). An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and the solution is $v = \dfrac{1}{x}\displaystyle\int 0\,dx = \dfrac{a}{x}$, that is $u' = \dfrac{a}{x}$. Integrate to get $u = \displaystyle\int \frac{a}{x}\,dx = a\ln x + b$. Thus we have $y = \frac{1}{x}\,u = \frac{1}{x}(a\ln x + b)$. We only need one more independent solution, so we take $a = 1$ and $b = 0$ to get $y_2 = \dfrac{\ln x}{x}$. Thus the general solution is $y = \dfrac{A}{x} + \dfrac{B\ln x}{x}$.

[5] **5:** Solve the DE $4y'' - 4y' + y = x\,e^x$.

Solution: The characteristic equation is $4r^2 - 4r + 1 = 0$, ,that is $(2r-1)^2 = 0$. The general solution to the homogeneous DE is $y = Ae^{x/2} + Bxe^{x/2}$. For a particular solution to the non-homogeneous DE we try $y = y_p = (a + bx)e^x$. Then $y' = (a + b + bx)e^x$ and $y'' = (a + 2b + bx)e^x$. Put these in the DE to get

$$\begin{aligned} x\,e^x &= 4y'' - 4y' + y \\ &= 4(a + 2b + bx)e^x - 4(a + b + bx)e^x + (a + bx)e^x \\ &= (a + 4b)e^x + bxe^x\,. \end{aligned}$$

Dividing by e^x gives $1x = (a + 4b) + bx$. Equate coefficients to get $b = 1$ and $a + 4b = 0$ so $a = -4$. Thus we obtain the particular solution $y_p = (x - 4)e^x$. The general solution to the given DE is

$$y = Ae^{x/2} + Bxe^{x/2} + (x - 4)e^x\,.$$

[5] **6:** An object of mass $m = 1$ kg is attached to a spring of spring-constant $k = 5$ N/m in a liquid where the damping-constant is $c = 2$ kg/s. A constant driving force of $F = 5$ N is applied to the object. Let $x(t)$ be the displacement of the object from the equilibrium position. Given that $x(0) = 0$ and $x'(0) = 0$, find the maximum value of x.

Solution: The displacement $x(t)$ satisfies the DE $mx'' + cx' + kx = F$. With the given values, this becomes $x'' + 2x' + 5x = 5$. The characteristic equation is $r^2 + 2r + 5 = 0$. Solve this to get $r = -1 \pm 2i$, so the general solution to the associated homogeneous DE is $y = Ae^{-t}\sin 2t + Be^{-t}\cos 2t$. For a particular solution we try $x = x_p = a$ (a constant), so $x' = 0$ and $x'' = 0$. We put these in the DE and get $5a = 5$, so $a = 1$, and so our particular solution is $x_p = 1$. The general solution to the given (non-homogeneous) DE is $x = Ae^{-t}\sin 2t + Be^{-t}\cos 2t + 1$, and then we have $x' = -Ae^{-t}\sin 2t + 2Ae^{-t}\cos 2t - Be^{-t}\cos 2t - 2Be^{-t}\sin 2t$. To get $x(0) = 0$ we need $B + 1 = 0$ so $B = -1$, and to get $x'(0) = 0$ we need $2A - B = 0$ so $A = \frac{1}{2}B = -\frac{1}{2}$. Thus the displacement is $x(t) = -\frac{1}{2}e^{-t}\sin 2t - e^{-t}\cos 2t + 1$ and the velocity is given by $x' = \frac{5}{2}e^{-t}\sin 2t$. The displacement is maximized when $x' = 0$, and we have $x'(t) = 0 \iff \sin 2t = 0 \iff 2t = 0, \pi, 2\pi, \cdots \iff t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \cdots$. Since $x(0) = 0$, $x\left(\frac{\pi}{2}\right) = 1 + e^{-\pi/2} > 1$, $x(\pi) = 1 - e^{-\pi} < 1$, $x\left(\frac{3\pi}{2}\right) = 1 + e^{-3\pi/2} > 1$, and so on (the displacement oscillates above and below the new equilibrium position 1 with smaller and smaller oscillations), the maximum displacement is $x\left(\frac{\pi}{2}\right) = 1 + e^{-\pi/2}$.

[5] **7:** Use the Power Series Method to find the solution of the IVP $y'' - 2x\,y' - 2y = 0$ with $y(0) = 1$, $y'(0) = 0$. First find a power series solution, then convert the power series to closed form.

Solution: Let $y = \sum_{n=0}^{\infty} a_n x^n$ so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Put these in the DE to get

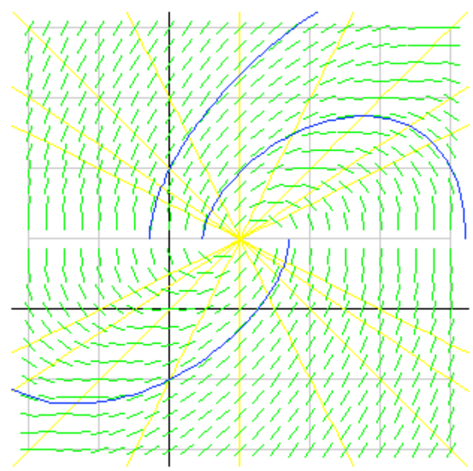
$$\begin{aligned} 0 = y'' - 2xy' - 2y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} 2ma_m x^m - \sum_{m=0}^{\infty} 2a_m x^m \\ &= \big(2a_2 - 2a_0\big) + \sum_{m=1}^{\infty} \big((m+2)(m+1)a_{m+2} - 2(m+1)a_m\big)x^m \,. \end{aligned}$$

All the coefficients must vanish, so we have $a_2 = a_0$ and for $m \geq 1$, $a_{m+2} = \dfrac{2(m+1)a_m}{(m+2)(m+1)} = \dfrac{2a_m}{m+2}$. To get $y(0) = 1$ and $y'(0) = 0$, we need $a_0 = 1$ and $a_1 = 0$, and then the recurrence formula gives $a_k = 0$ for k odd and $a_2 = 1$, $a_4 = \frac{2}{4}$, $a_6 = \frac{2^2}{2\cdot 6}$, and in general $a_{2n} = \frac{2^n}{2\cdot 4\cdot 6\cdots(2n)} = \frac{1}{n!}$. Thus the solution is $y = \sum_{n=1}^{\infty} \dfrac{x^{2n}}{n!} = e^{x^2}$.

[5] **8:** Consider the DE $y' = \dfrac{y - x}{y - 1}$.

(a) Sketch the direction field for the given DE along with the solution curves through each of the points $(0, -1)$, $(0, 2)$ and $(4, 2)$.

Solution: The isocline $y' = m$ is given by $\dfrac{y - x}{y - 1} = m$, that is $y - x = my - m$ or equivalently $x + (m-1)y = m$. This is the line with slope $-\frac{1}{m-1}$ and x-intercept at $(m, 0)$. Some of these lines are shown in yellow, the slope field is shown in green, and the solution curves are shown in blue. Note that the solution curves do not pass the line $y = 1$ along which the slope is undefined.



(b) Use Euler's Method with step size $\Delta x = 1$ to approximate the value of $f(3)$ where $y = f(x)$ is the solution to the given DE with $y(0) = 2$.

Solution: The values of x_k and y_k and $F(x_k, y_k) = \dfrac{y_k - x_k}{y_k - 1}$ are tabulated below.

k	x_k	y_k	$\frac{y_k - x_k}{y_k - 1}$
0	0	2	2
1	1	4	1
2	2	5	$\frac{3}{4}$
3	3	$\frac{23}{4}$	

Thus we have $f(3) \cong y_3 = \frac{23}{4}$.