MATH 218 Differential Equations, Solutions to the Final Exam, Fall 2008

[10] **1:** (a) Solve the IVP $y' = xy^2$ with $y(1) = 2$.

Solution: This DE is separable since we can write it as $\dfrac{y'}{y^2} = x$. Integrate both sides (with respect to x) to get $-\dfrac{1}{y} = \frac{1}{2}x^2 + a$. To get $y(1) = 2$ we need $-\frac{1}{2} = \frac{1}{2} + a$ so $a = -1$. Thus the solution is given by $-\dfrac{1}{y} = \frac{1}{2}x^2 - 1 = \dfrac{x^2-2}{2}$, that is $y = \dfrac{2}{2-x^2}$.

(b) Solve the IVP $y' + y = y^3$ with $y(0) = \frac{1}{2}$.

Solution: This is a Bernoulli DE. We make the substitution $u = y^{-2}$ so $u' = -2y^{-3}y'$. Multiply both sides of the DE by $-2y^{-3}$ to get $-2y^{-3}y' - 2y^{-2} = -2$, that is $u' - 2u = -2$. This DE is linear for $u = u(x)$. An integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$ and the solution is $u = e^{2x}\displaystyle\int -2e^{-2x}\,dx = e^{2x}\big(e^{-2x} + a\big) = 1 + ae^{2x}$. Replace u by y^{-2} to get $y^{-2} = 1 + ae^{2x}$. To get $y(0) = \frac{1}{2}$ we need $4 = 1 + a$ so $a = 3$. Thus the solution is given by $y^{-2} = 1 + 3e^{2x}$, that is $y = \dfrac{1}{\sqrt{1+3e^{2x}}}$.

[10] **2:** A tank, in the shape of a rectangular box which is 4 m tall and has a square base with sides of length 1 m, initially contains 1 m^3 of water. Water pours in at the rate of 4 L/s $= \frac{1}{250}$ m^3/s. Water pours out through a hole of area 5 $\text{cm}^2 = \frac{1}{2000}$ m^2 in the base of the tank at a speed of $4\sqrt{y}$ m/s, where y is the depth of the water in the tank, in meters. Find the time at which the water reaches a depth of $\frac{9}{4}$ meters.

Solution: Let $V = V(t)$ be the volume of water in the tank. Note that $V = 1 \times 1 \times y = y$ so $V' = y'$. Also, we have $V' = \frac{1}{250} - \frac{1}{2000} \cdot 4\sqrt{y} = \frac{1}{250} - \frac{1}{500}\, y^{1/2}$. Thus $y = y(t)$ satisfies the DE $y' = \frac{1}{250} - \frac{1}{500}\, y^{1/2}$, that is

$$500\, y' = 2 - \sqrt{y}$$

with $y(0) = 1$. This DE is separable since we can write it as $\dfrac{500\, y'}{2 - \sqrt{y}} = 1$. Integrate both sides (with respect to t) to get

$$\int \frac{500\, dy}{2 - \sqrt{y}} = \int 1\, dt\,.$$

To solve the integral on the left, we make the substitution $u = 2 - \sqrt{y}$, $\sqrt{y} = 2 - u$, $y = 4 - 4u + u^2$, $dy = (2u - 4)du$ to get

$$\begin{aligned}\int \frac{500\, dy}{2-\sqrt{y}} &= \int \frac{500(2u-4)du}{u} = \int 1000 - \frac{2000}{u}\, du = 1000\, u - 2000 \ln u + a \\ &= 1000(2 - \sqrt{y}) - 2000\ln(2 - \sqrt{y}) + a\,.\end{aligned}$$

Thus the solution to the DE is given by $1000(2-\sqrt{y}) - 2000\ln(2-\sqrt{y}) = t + b$. To get $y(0) = 1$ we need $1000 = b$, so the solution is given by $t = 1000(2-\sqrt{y}) - 2000\ln(2-\sqrt{y}) - 1000 = 1000\big(1 - \sqrt{y} - 2\ln(2-\sqrt{y})\big)$. When $y = \frac{9}{4}$ we have $t = 1000\big(1 - \frac{3}{2} - 2\ln\frac{1}{2}\big) = 500(4\ln 2 - 1)$.

[10] **3:** Solve the DE $y'' - y' - 2y = 4t^2$.

Solution: The Characteristic equation is $r^2 - r - 2 = 0$. Solve this to get $r = -1, 2$. The solution to the associated homogeneous DE is $y = Ae^{-t} + Be^{2t}$. To find a particular solution to the given (non-homogeneous) DE we try $y_p = a+bt+ct^2$. Note that $y_p{}' = b+2ct$ and $y_p{}'' = 2c$. Put this in the DE to get

$$\begin{aligned} 4t^2 &= y_p{}'' - y_p{}' - 2y_p \\ &= 2c - (b + 2ct) - 2(a + bt + ct^2) \\ &= (2c - b - 2a) - (2b + 2c)t - (2c)t^2 \,. \end{aligned}$$

Equate coefficients to get $2c - b - 2a = 0$, $2b + 2c = 0$ and $-2c = 4$. Solve these three equations to get $a = -3$, $b = 2$ and $c = -2$. Thus we obtain the particular solution $y_p = -3 + 2t - 2t^2$. The general solution to the given DE is $y = Ae^{-t} + Be^{2t} - 3 + 2t - 2t^2$.

[10] **4:** Solve the DE $x^2y'' - 2xy + 2y = x + 2$, given that $y = x$ and $y = x^2$ are solutions to the associated homogeneous DE.

Solution: We can write the DE in the form $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{x+2}{x^2}$. Using the method of variation of parameters, we try a solution of the form $y = xu + x^2v$ for some functions $u = u(x)$ and $v = v(x)$ such that $xu' + x^2v' = 0$ (1). Putting this into the DE will give us $u' + 2xv' = \frac{x+2}{x^2}$ (2). We can write these two equations as $\begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{x+2}{x^2} \end{pmatrix}$. We have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{x+2}{x^2} \end{pmatrix} = \frac{1}{x^2}\begin{pmatrix} 2x & -x^2 \\ -1 & x \end{pmatrix}\begin{pmatrix} 0 \\ \frac{x+2}{x^2} \end{pmatrix} = \begin{pmatrix} -\frac{x+2}{x^2} \\ \frac{x+2}{x^3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{x} - \frac{2}{x^2} \\ \frac{1}{x^2} + \frac{2}{x^3} \end{pmatrix}.$$

so $u = \displaystyle\int -\frac{1}{x} - \frac{2}{x^2}\, dx = -\ln x + \frac{2}{x}$ and $v = \displaystyle\int \frac{1}{x^2} + \frac{2}{x^3}\, dx = -\frac{1}{x} - \frac{1}{x^2}$. Thus a particular solution to the DE is $y_p = xu + x^2v = -x\ln x + 2 - x - 1 = 1 - x - x\ln x$. The general solution to the DE is $y = ax + bx^2 + 1 - x - x\ln x$.

[10] **5:** Use power series to solve the DE $(2+x^2)y''+4xy'+2y=0$. Express your answer in closed form.

Solution: Let $y=\sum_{n=0}^{\infty}a_nx^n$. Then $y'=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$. Put these in the DE to get

$$\begin{aligned} 0&=(2+x^2)y''+4xy'+2y\\ &=\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=1}^{\infty}4na_nx^n+\sum_{n=0}^{\infty}2a_nx^n\\ &=\sum_{m=0}^{\infty}2(m+2)(m+1)a_{m+2}x^m+\sum_{m=0}^{\infty}\big(m(m-1)+4m+2\big)a_mx^m\,. \end{aligned}$$

All coefficients must vanish, so for all $m\geq 0$ we have

$$\begin{aligned} 2(m+2)(m+1)a_{m+2}&=-(m^2-m+4m+2)a_m=-(m^2+3m+2)a_m\\ &=-(m+2)(m+1)a_m\,, \end{aligned}$$

that is $a_{m+2}=-\frac{1}{2}a_m$. If we take $a_0=1$ and $a_1=0$ then we get $a_3=a_5=a_7=\cdots=0$ and $a_2=-\frac{1}{2}$, $a_4=\frac{1}{2^2}$, $a_6=-\frac{1}{2^3}$, and so on so we obtain the solution

$$y_1=1-\tfrac{1}{2}x^2+\tfrac{1}{2^2}x^4-\tfrac{1}{2^3}x^6+\cdots=\frac{1}{1+\frac{1}{2}x^2}=\frac{2}{2+x^2}\,.$$

If we take $a_0=0$ and $a_1=1$ then we get $a_2=a_4=a_6=\cdots=0$ and $a_3=-\frac{1}{2}$, $a_5=\frac{1}{2^2}$, $a_7=-\frac{1}{2^3}$ and so on, so we obtain the solution

$$y_2=x-\tfrac{1}{2}x^3+\tfrac{1}{2^2}x^5-\tfrac{1}{2^3}x^7+\cdots=xy_1=\frac{2x}{2+x^2}\,.$$

Thus the general solution to the DE is $y=\dfrac{a+bx}{2+x^2}$.

[10] **6:** Use Laplace transforms to solve the IVP $y' + y = g(t)$ with $y(0) = 1$, where

$$g(t) = \begin{cases} t-2 \text{ , for } 0 \le t \le 3, \\ \quad 1 \quad \text{, for } 3 \le t \end{cases} .$$

Express your answer in piecewise form.

Solution: Note that $g(t) = -2 + t - (t-3)H(t-3)$ so taking the Laplace transform on both sides off the DE gives

$$\begin{aligned} -1 + sY + Y &= -\frac{2}{s} + \frac{1}{s^2} - \frac{e^{-3s}}{s^2} \\ (s+1)Y &= 1 - \frac{2}{s} + \frac{1}{s^2} - \frac{e^{-3s}}{s^2} = \frac{s^2 - 2s + 1 - e^{-3s}}{s^2} \\ Y &= \frac{s^2 - 2s + 1 - e^{-3s}}{s^2(s+1)} . \end{aligned}$$

To get $\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} = \frac{s^2 - 2s + 1}{s^2(s+1)}$ we need $As(s+1) + B(s+1) + Cs^2 = s^2 - 2s + 1$. Equate coefficients to get $A + C = 1$, $A + B = -2$ and $B = 1$. Solve these to get $A = -3$, $B = 1$ and $C = 4$. Also, to get $\frac{D}{s} + \frac{E}{s^2} + \frac{F}{s+1} = \frac{1}{s^2(s+1)}$ we need $Ds(s+1) + E(s+1) + Fs^2 = 1$ so $D + F = 0$, $D + E = 0$ and $E = 1$, and so $D = -1$, $E = 1$ and $F = 1$. Thus we have

$$Y = \left(-\frac{3}{3} + \frac{1}{s^2} + \frac{4}{s+1} \right) - \left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \right) e^{-3s} .$$

Take the inverse Laplace transform to get

$$\begin{aligned} y &= \big(-3 + t + 4e^{-t} \big) - \big(-1 + (t-3) + e^{-(t-3)} \big) H(t-3) \\ &= \big(-3 + t + 4e^{-t} \big) + \big(4 - t - e^3 e^{-t} \big) H(t-3) \\ &= \begin{cases} -3 + t + 4e^{-t} & \text{, for } 0 \le t \le 3 \\ 1 + (4 - e^3) e^{-t} & \text{, for } 3 \le t \end{cases} \end{aligned}$$

[10] **7:** Let $x(t)$ be the position, in meters at time t seconds, of an object of mass $m = \frac{1}{4}$ kg which is attached to a spring of spring-constant $k = 2$ N/m in a liquid where the damping-constant is $c = 1$ kg/s. The object is released from the position $x(0) = 1$ with $x'(0) = 0$, then it is struck once with a hammer at time $t = \frac{\pi}{2}$ increasing its momentum by 1 kg m/s. Use Laplace transforms to find $x(t)$, then find the velocity at time $t = \pi$.

Solution: The position $x(t)$ satisfies the DE $\frac{1}{4}x'' + x' + 2x = \delta\left(t - \frac{\pi}{2}\right)$, that is

$$x'' + 4x' + 8x = 4\delta\left(t - \tfrac{\pi}{2}\right)\,,$$

with $x(0) = 1$ and $x'(0) = 0$. Take the Laplace transform to get

$$\begin{aligned}(-s + s^2X) + 4(-1 + sX) + 8(X) &= 4e^{-\pi s/2}\\ (s^2 + 4s + 8)X &= s + 4 + 4e^{-\pi s/2}\\ X = \frac{s + 4 + 4e^{-\pi s/2}}{s^2 + 4s + 8} &= \frac{(s+2) + 2 + 4e^{-\pi/2}}{(s+2)^2 + 4}\,.\end{aligned}$$

Take the inverse Laplace transform to get

$$\begin{aligned}x &= e^{-2t}\cos 2t + e^{-2t}\sin 2t + 2e^{-2\left(t - \frac{\pi}{2}\right)}\sin\left(2\left(t - \tfrac{\pi}{2}\right)\right)H\left(t - \tfrac{\pi}{2}\right)\\ &= e^{-2t}\cos 2t + e^{-2t}\sin 2t - 2e^{\pi}e^{-2t}\sin 2t \cdot H\left(t - \tfrac{\pi}{2}\right)\\ &= \begin{cases} e^{-2t}\cos 2t + e^{-2t}\sin 2t & \text{, for } 0 \le t \le \dfrac{\pi}{2}\\ e^{-2}\cos 2t + (1 - 2e^{\pi})e^{-2t}\sin 2t & \text{, for } \dfrac{\pi}{2} \le t\,. \end{cases}\end{aligned}$$

For $t > \frac{\pi}{2}$ we have

$$x' = -2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t - 2(1 - 2e^{\pi})e^{-2t}\sin 2t + 2(1 - 2e^{\pi})e^{-2t}\cos 2t$$

so in particular $x'(\pi) = -2e^{-2\pi} + 2(1 - 2e^{\pi})e^{-2\pi} = -\dfrac{4}{e^{2\pi}}$.

[10] **8:** Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Solve the system.

Solution: Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. Then

$$\left|A - rI\right| = \begin{vmatrix} 1-r & -2 \\ 2 & 1-r \end{vmatrix} = r^2 - 2r + 5 = (r-1)^2 + 4\,.$$

The eigenvalues are $r = 1 \pm 2i$. When $r = 1 + 2i$,

$$A - rI = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

so an eigenvalue is $u = \begin{pmatrix} i \\ 1 \end{pmatrix}$. This gives the complex solution

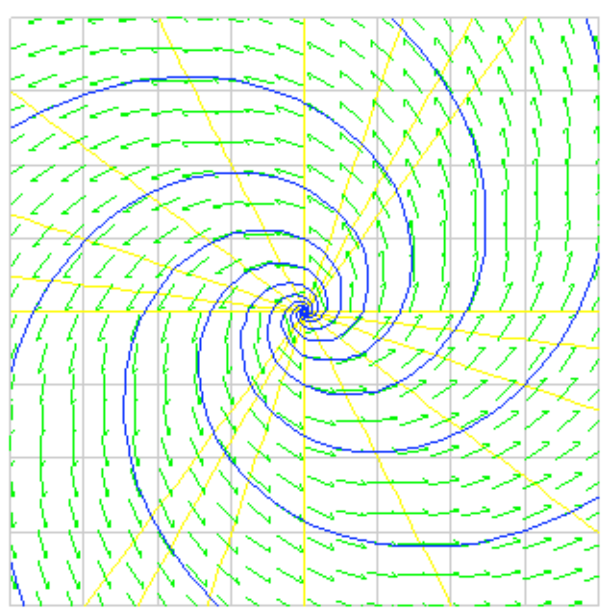
$$\begin{pmatrix} z \\ w \end{pmatrix} = e^{(1+2i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \left(\cos 2t + i \sin 2t\right) \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \begin{pmatrix} -\sin 2t + i \cos 2t \\ \cos 2t + i \sin 2t \end{pmatrix}.$$

Using the real and imaginary parts, we obtain the general solution to the system:

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^t \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + be^t \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}.$$

(b) Sketch the phase portrait for the system.

Solution: The isoclines are given by $m = \frac{y'}{x'} = \frac{2x+y}{x-2y}$, that is $mx - 2my = 2x + y$, or equivalently $y = \frac{m-2}{2m+1}$. This is the line through $(0,0)$ with slope $\frac{m-2}{2m+1}$. The isoclines $m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$ are shown in yellow, the slope field is shown in green, and some solution curves are shown in blue.



[10] **9:** Find the solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t/y \\ -t/x \end{pmatrix}$ with $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

Solution: First we solve the DE $\frac{dy}{dx} = \frac{y'}{x'} = \frac{-t/x}{t/y} = -\frac{y}{x}$ for $y = y(x)$. This DE is separable since we can write it as $\frac{y'}{y} = -\frac{1}{x}$. Integrate both sides (with respect to x) to get $\ln y = -\ln x + a$. Put in $t = 0$, $x = \frac{1}{2}$ and $y = 1$ to get $1 = \frac{b}{1/2}$ so $b = \frac{1}{2}$. Thus we have $y = \frac{\frac{1}{2}}{x} = \frac{1}{2x}$. Now put this into the DE $x' = t/y$ to get $x' = 2xt$. This DE is separable since we can write it as $\frac{x'}{x} = 2t$. Integrate both sides (with respect to t) to get $\ln x = t^2 + b$. Put in $t = 0$ and $x = \frac{1}{2}$ to get $\ln \frac{1}{2} = c$, so the solution $x = x(t)$ is given by $\ln x = t^2 + \ln \frac{1}{2}$, that is $x = \frac{1}{2}e^{t^2}$. Note that $x' = te^{t^2}$, so from the DE $x' = y/t$ we obtain $y = \frac{t}{x'} = \frac{t}{te^{x^2}} = e^{-t^2}$. Thus the solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{t^2} \\ e^{-t^2} \end{pmatrix} .$$

[10] **10:** Tank A initially contains 3 L of pure water, and tank B initially contains 2 L of pure water. Brine, with a salt concentration of 3 g/L enters tank A at a rate of 4 L/hr. Brine is pumped from tank A to tank B at 6 L/hr, and brine is pumped back from tank B to tank A at 2 L/hr. Also, brine drains from tank B at 4 L/hr. Both tanks are kept well mixed at all times. Find the amount of salt in each tank at time t.

Solution: Note that the volumes in both tanks remain constant. Let $x(t)$ and $y(t)$ be the amounts of salt in tanks A and B respectively. Then we have

$$x' = 4 \cdot 3 - 6 \cdot \frac{x}{3} + 2 \cdot \frac{y}{2} = -2x + y + 12$$

and

$$y' = 6 \cdot \frac{x}{3} - 2 \cdot \frac{y}{2} - 4 \cdot \frac{y}{2} = 2x - 3y\,,$$

that is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$. Then

$$\left|A - rI\right| = \begin{vmatrix} -2-r & 1 \\ 2 & -3-r \end{vmatrix} = r^2 + 5r + 4 = (r+1)(r+4)$$

so the eigenvalues are $r = -1, -4$. For $r = -1$ we have

$$A - rI = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

so an eigenvalue is $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. When $r = -4$ we have

$$A - rI = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

so an eigenvalue is $v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Thus the general solution to the associated homogeneous system of DEs is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

To find a particular solution to the non-homogeneous DE we try $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$. Put this into the system of DEs to get

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

so

$$\begin{pmatrix} c \\ d \end{pmatrix} = -\begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}.$$

Thus the general solution to the system of DEs is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 9 \\ 6 \end{pmatrix}.$$

To get $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we need $a - b = 9$ and $a + 2b = 6$. Solve these two equations to get $a = -8$ and $b = 1$. Thus the amount of salt in each tank is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = -8e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 9 \\ 6 \end{pmatrix},$$

or equivalently, $x(t) = 9 - 8e^{-t} - e^{-4t}$ and $y(t) = 6 - 8e^{-t} + 2e^{-4t}$.