Name (print): _____

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ID Number: _____

# MATH 218, Differential Equations for Engineers

## Final Examination, Fall Term, 2008

## University of Waterloo

Instructor: Stephen New

Date: December 11, 2008

Time: 12:30-3:00 pm

| Question | Mark |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 | /10 |
| 8 | /10 |
| 9 | /10 |
| 10 | /10 |
| Best 9 of 10 | /90 |

Instructions:

1. Place your name, signature and ID number in the spaces provided at the top of this page.

2. This test contains 11 pages, including this one.

3. No calculators are allowed.

4. Answer any 9 of the 10 questions. If you answer all 10, then the best 9 will be counted.

5. Provide full explanations with all your solutions.

[10] **1:** (a) Solve the IVP $y' = xy^2$ with $y(1) = 2$.

(b) Solve the IVP $y' + y = y^3$ with $y(0) = \frac{1}{2}$.

[10] **2:** A tank, in the shape of a rectangular box which is 4 m tall and has a square base with sides of length 1 m, initially contains 1 $m^3$ of water. Water pours in at the rate of 4 L/s $= \frac{1}{250}$ $m^3$/s. Water pours out through a hole of area 5 $cm^2 = \frac{1}{2000}$ $m^2$ in the base of the tank at a speed of $4\sqrt{y}$ m/s, where $y$ is the depth of the water in the tank, in meters. Find the time at which the water reaches a depth of $\frac{9}{4}$ meters.

[10] **3:** Solve the DE $y'' - y' - 2y = 4t^2$.

[10] **4:** Solve the DE $x^2y'' - 2xy + 2y = x + 2$, given that $y = x$ and $y = x^2$ are solutions to the associated homogeneous DE.

[10] **5:** Use power series to solve the DE $(2 + x^2)y'' + 4xy' + 2y = 0$. Express your answer in closed form.

[10] **6:** Use Laplace transforms to solve the IVP $y' + y = g(t)$ with $y(0) = 1$, where

$$g(t) = \begin{cases} t - 2 \text{ , for } 0 \leq t \leq 3, \\ 1 \text{ , for } 3 \leq t \end{cases} .$$

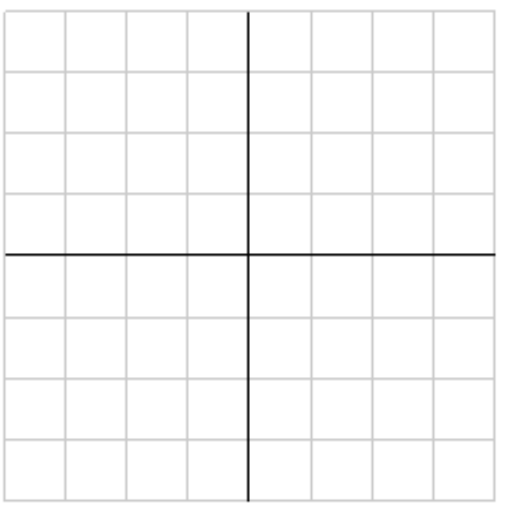
Express your answer in piecewise form.

[10] **7:** Let $x(t)$ be the position, in meters at time $t$ seconds, of an object of mass $m = \frac{1}{4}$ kg which is attached to a spring of spring-constant $k = 2$ N/m in a liquid where the damping-constant is $c = 1$ kg/s. The object is released from the position $x(0) = 1$ with $x'(0) = 0$, then it is struck once with a hammer at time $t = \frac{\pi}{2}$ increasing its momentum by 1 kg m/s. Use Laplace transforms to find $x(t)$, then find the velocity at time $t = \pi$.

[10] **8:** Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Solve the system.

(b) Sketch the phase portrait for the system.



[10] **9:** Find the solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t/y \\ -t/x \end{pmatrix}$ with $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

[10] **10:** Tank $A$ initially contains 3 L of pure water, and tank $B$ initially contains 2 L of pure water. Brine, with a salt concentration of 3 g/L enters tank $A$ at a rate of 4 L/hr. Brine is pumped from tank $A$ to tank $B$ at 6 L/hr, and brine is pumped back from tank $B$ to tank $A$ at 2 L/hr. Also, brine drains from tank $B$ at 4 L/hr. Both tanks are kept well mixed at all times. Find the amount of salt in each tank at time $t$.