MATH 218 Differential Equations, Solutions to Assignment 9

1: Use Laplace transforms to solve the following IVPs.

(a) $y' - 2y = e^{2t}$ with $y(0) = 3$.

Solution: Take the Laplace transform of both sides to get

$$\begin{aligned} -3 + sY - 2Y &= \frac{1}{s-2} \\ (s-2)Y &= \frac{1}{s-2} + 3 = \frac{3s-5}{s-2} \\ Y &= \frac{3s-5}{(s-2)^2}\,. \end{aligned}$$

To get $\dfrac{3s-5}{(s-2)^2} = \dfrac{A}{s-2} + \dfrac{B}{(s-2)^2}$ we need $A(s-2) + B = 3s - 5$. Equate coefficients to get $A = 3$ and $-2A + B = -5$ so $B = 1$. Thus $Y = \dfrac{3}{s-3} + \dfrac{1}{(s-2)^2}$. Take the inverse Laplace transform of bath sides to get

$$y = 3e^{2t} + te^{2t}\,.$$

(b) $y'' - 6y' + 13y = 0$ with $y(0) = 2$, $y'(0) = 0$.

Solution: Take the Laplace transform of both sides to get

$$\begin{aligned} (-2s + s^2Y) - 6(-2 + sY) + 13Y &= 0 \\ (s^2 - 6s + 13)Y &= 2s - 12 \\ Y = \frac{2s-12}{s^2 - 6s + 13} = \frac{2s-12}{(s-3)62 + 4} &= \frac{2(s-3)}{(s-3)^2 + 4} - \frac{6}{(s-3)^2 + 4}\,. \end{aligned}$$

Take the inverse Laplace transform of both sides to get

$$y = 2e^{3t}\cos(2t) - 3e^{3t}\sin(2t)\,.$$

2: Use Laplace transforms to solve the following IVPs.

(a) $y'' + 2y' + y = e^{-t}$ with $y(0) = 1$, $y'(0) = 2$.

Solution: Take the Laplace transform of both sides to get

$$\begin{aligned}(-2 - s + s^2Y) + 2(-1 + sY) + Y &= \frac{1}{s+1}\\ (s^2 + 2s + 1)Y &= \frac{1}{s+1} + s + 4 = \frac{s^2 + 5s + 5}{s+1}\\ Y &= \frac{s^2 + 5s + 5}{(s+1)(s^2 + 2s + 1)} = \frac{s^2 + 5s + 5}{(s+1)^3}\end{aligned}$$

To get $\frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} = \frac{s^2+5s+5}{(s+1)^3}$ we need $A(s+1)^2 + B(s+1) + C = s^2 + 5s + 5$. Equate coefficients to get $A = 1$, $2A + B = 5$, $A + B + C = 5$. Solve these to get $A = 1$, $B = 3$ and $C = 1$. Thus $Y = \frac{1}{s+1} + \frac{3}{(s+1)^2} + \frac{1}{(s+1)^3}$. Take the inverse Laplace transform to get

$$y = e^{-}t(1 + 3t + \tfrac{1}{2}t^2)\,.$$

(b) $2y'' - 3y' - 2y = 5e^{2t}$ with $y(0) = 1$, $y'(0) = 2$.

Solution: Take the Laplace transform of both sides to get

$$\begin{aligned}2(-2 - s + s^2Y) - 3(-1 + sY) - 2Y &= \frac{5}{s-2}\\ (2s^2 - 3s - 2)Y &= \frac{5}{s-2} + 2s + 1 = \frac{2s^2 - 3s + 3}{s-2}\\ Y &= \frac{2s^2 - 3s + 3}{(s-2)(2s^2 - 3s - 2)} = \frac{2s^2 - 3s + 3}{(s-2)^2(2s+1)}\,.\end{aligned}$$

To get $\frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{2s+1} = \frac{2s^2-3s+3}{(s-2)^2(2s+1)}$ we need $A(s-2)(2s+1) + B(2s+1) + C(s-2)^2 = 2s^2 - 3s + 3$. Equate coefficients and put in $s = 2$ and $s = -\frac{1}{2}$ to get $2A + C = 2$, $-3A + 2B - 4C = -3$, $-2A + B + 4C = 3$, $5B = 5$ and $\frac{25}{4}C = 5$. Solve these to get $A = \frac{3}{5}$, $B = 1$ and $C = \frac{4}{5}$. Thus we have $Y = \frac{\frac{3}{5}}{s-2} + \frac{1}{(s-2)^2} + \frac{\frac{4}{5}}{(2s+1)} = \frac{\frac{3}{5}}{s-2} + \frac{1}{(s-2)^2} + \frac{\frac{2}{5}}{\left(s+\frac{1}{2}\right)}$. Take the inverse Laplace transform to get

$$y = \tfrac{3}{5}e^{2t} + te^{2t} + \tfrac{2}{5}e^{-t/2}\,.$$

3: Use Laplace transforms to solve the following IVPs.

(a) $y'' + 3y' + 2y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$, where $g(t) = \begin{cases} 2 & \text{, if } 0 \le t \le 2, \\ 6 - 2t & \text{, if } 2 \le t \le 3, \\ 0 & \text{, if } 3 \le t\,. \end{cases}$

Solution: Note that $g(t) = 2H(t) - 2(t-2)H(t-2) + 2(t-3)H(t-3)$. Take the Laplace transform of both sides of the DE to get

$$\begin{aligned} s^2Y + 3sY + 2Y &= \frac{2}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} = \frac{2s - 2e^{-2s} + 2e^{-3s}}{s^2} \\ Y &= \frac{2s - 2e^{-2s} + 2e^{-3s}}{s^2(s^2+3s+2)} = \frac{2s - 2e^{-2s} + 2e^{-3s}}{s^2(s+1)(s+2)}\,. \end{aligned}$$

Note that $\dfrac{2s}{s^2(s+1)(s+2)} = \dfrac{2}{s(s+1)(s+2)} = \dfrac{1}{s} - \dfrac{2}{s+1} + \dfrac{1}{s+2}$. Also, to get $\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+1} + \dfrac{D}{s+2} = \dfrac{2}{s^2(s+1)(s+2)}$ we need $As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1) = 2$. Equate coefficients and put in $s = -1$ and $s = -2$ to get $A + C + D = 0$, $3A + B + 2C + D = 0$, $2A + 3B = 0$, $2B = 2$, $C = 2$ and $D = -\frac{1}{2}$. Solve these to get $A = -\frac{3}{2}$, $B = 1$, $C = 2$ and $D = -\frac{1}{2}$. Thus we find that

$$Y = \left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right) + \left(\frac{\frac{3}{2}}{s} - \frac{1}{s^2} - \frac{2}{s+1} + \frac{\frac{1}{2}}{s+2}\right)\left(e^{-2s} - e^{-3s}\right)\,.$$

Take the inverse Laplace transform of both sides to get

$$\begin{aligned} y = \left(1 - 2e^{-t} + e^{-2t}\right) &+ \left(\tfrac{7}{2} - t - 2e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-2)}\right)H(t-2) \\ &- \left(\tfrac{9}{2} - t - 2e^{-(t-3)} + \tfrac{1}{2}e^{-2(t-3)}\right)H(t-3)\,. \end{aligned}$$

Alternatively, we can write this in piecewise form as

$$y = \begin{cases} 1 - 2e^{-t} + e^{-2t} & \text{, for } 0 \le t \le 2 \\ \frac{9}{2} - t - 2(1+e^2)e^{-t} + \left(1 + \frac{1}{2}e^4\right)e^{-2t} & \text{, for } 2 \le t \le 3 \\ 2(e^3 - e^2 - 1)e^{-t} - \left(\frac{1}{2}e^6 - \frac{1}{2}e^4 - 1\right)e^{-2t} & \text{, for } 3 \le t \end{cases}$$

(b) $2y''' - 3y'' + 0y' + y = 0$ with $y(0) = 1$, $y'(0) = -3$ and $y''(0) = 2$.

Solution: Take the Laplace transform on both sides to get

$$\begin{aligned} 2(-2 + 3s - s^2 + s^3Y) - 3(3 - s + s^2Y) + Y &= 0 \\ (2s^3 - 3s^2 + 1)Y &= 13 - 9s + 2s^2 \\ Y = \frac{2s^2 - 9s + 13}{2s^3 - 3s^2 + 1} = \frac{2s^2 - 9s + 13}{(s-1)(2s^2 - s - 1)} &= \frac{2s^2 - 9s + 13}{(s-1)^2(2s+1)}\,. \end{aligned}$$

To get $\dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{2s+1} = \dfrac{2s^2 - 9s + 13}{(s-1)^2(2s+1)}$ we need $A(s-1)(2s+1) + B(2s+1) + C(s-1)^2 = 2s^2 - 9s + 13$. Equate coefficients and put in $s = 1$ and $s = -\frac{1}{2}$ to get $2A + C = 2$, $-A + 2B - 2C = -9$, $-A + B + C = 13$, $3B = 6$ and $\frac{9}{4}C = 18$. Solve these to get $A = -3$, $B = 2$ and $C = 8$. Thus we have $Y = -\dfrac{3}{s-1} + \dfrac{2}{(s-1)^2} + \dfrac{4}{s + \frac{1}{2}}$. Take the inverse Laplace transform to get

$$y = -3e^t + 2te^t + 4e^{-t/2}\,.$$

4: A tank initially contains 20 L of pure water. Brine pours in at the rate of 4 L/min. The concentration (in g/L) of this brine at time t minutes is given by

$$c(t) = \begin{cases} t & \text{, if } 0 \le t \le 5, \\ 10 - t & \text{, if } 5 \le t \le 10, \\ 0 & \text{, if } 10 \le t\,. \end{cases}$$

The tank is kept well mixed, and brine drains from the tank at 4 L/min. Use Laplace transforms to find the amount of salt $y(t)$ in the tank at time t, and in particular, find the time at which the amount of salt reaches its maximum.

Solution: The amount of salt $y(t)$ satisfies the IVP $y'(t) = 4\,c(t) - \dfrac{4y}{20}$ with $y(00$, that is

$$5y' + y = 20c(t) = 20\big(tH(t) - 2(t-5)H(t-5) + (t-10)H(t-10)\big)$$

with $y(0) = 0$. Take the Laplace transform of both sides to get

$$5sY + Y = 20\left(\frac{1}{s^2} - \frac{2e^{-5s}}{s^2} + \frac{e^{-10s}}{s^2}\right)$$
$$Y = \frac{20\left(1 - 2e^{-5s} + 10e^{-10s}\right)}{s^2(5s+1)}\,.$$

To get $\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{5s+1} = \dfrac{1}{s^2(5s+1)}$ we need $As(5s+1) + B(5s+1) + Cs^2 = 1$. Equate coefficients to get $5A + C = 0$, $A + 5B = 0$ and $B = 1$. Solve these to get $A = -5$, $B = 1$ and $C = 25$. Thus we have

$$Y = 20\left(-\tfrac{5}{s} + \tfrac{1}{s^2} + \tfrac{5}{s+\frac{1}{5}}\right)\left(1 - 2e^{-5s} + e^{-10s}\right)\,.$$

Take the inverse Laplace transform to get

$$\begin{aligned} y &= 20\Big((-5 + t + 5e^{-t/5}) - 2(-10 + t + 5e^{-(t-5)/5})H(t-5) + (-15 + t + 5e^{-(t-10)/5})\Big) \\ &= \begin{cases} 20\big(-5 + t + 5e^{-t/5}\big) & \text{, for } 0 \le t \le 5 \\ 20\left(15 - t - (10e - 5)e^{-t/5}\right) & \text{, for } 5 \le t \le 10 \\ 20\left(5e^2 - 10e + 5)e^{-t/5}\right) & \text{, for } 10 \le t \end{cases} \end{aligned}$$

Taking the derivative, we get

$$y' = \begin{cases} 20\left(1 - e^{-t/5}\right) & \text{, for } 0 \le t \le 5 \\ 20\big(-1 + (2e-1)e^{-t/5}\big) & \text{, for } 5 \le t \le 10 \\ 20\big(-(e^2 - 2e + 1)e^{-t/5}\big) & \text{, for } 10 \le t \end{cases}$$

We see that $y' \ge 0$ for $0 \le t \le 5$ and $y' \le 0$ for $10 \le t$, For $5 \le t \le 10$, note that $y'(5) > 0$ and $y'(10) < 0$ and we have $y'(t) = 0 \iff (2e-1)e^{-t/5} = 1 \iff e^{-t/5} = \frac{1}{2e-1} \iff e^{t/5} = 2e - 1 \iff \frac{1}{5}t = \ln(2e-1) \iff t = 5\ln(2e-1)$. Thus $y(t)$ attains its maximum value when $t = 5\ln(2e-1)$.

5: An object of mass $m = 1$ kg is attached to a spring of spring-constant $k = 5$ N/m in a liquid where the damping constant is $c = 2$ kg/s. The object is initially at rest at the equilibrium position, then it is struck repeatedly with a hammer, first at time $t = 0$, and then again every π seconds. Each strike of the hammer imparts an impulse which increases the object's momentum by 1 kg m/s^2. Determine the motion of the object, and in particular, find the asymptotic (or eventual) velocity of the object just after each strike.

Solution: The position $y(t)$ of the object satisfies the IVP

$$y'' + 2y' + 5y = \sum_{n=0}^{\infty} \delta(t - \pi n)$$

with $y(0) = 1$. Take the Laplace transform to get

$$s^2Y + 2sY + 5Y = \sum_{n=0}^{\infty} e^{-n\pi\, s}$$

$$Y = \sum_{n=0}^{\infty} \frac{e^{-n\pi\, s}}{s^2 + 2s + 5} = \sum_{n=0}^{\infty} \frac{e^{-n\pi\, s}}{(s+1)^2 + 4}$$

Take the inverse Laplace transform to get

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \tfrac{1}{2} e^{-(t-n\pi)} \sin(2t) H(t - n\pi) \\ &= \sum_{n=0}^{\infty} \tfrac{1}{2} e^{n\pi} e^{-t} \sin(2t) H(t - n\pi) \\ &= \begin{cases} \tfrac{1}{2} e^{-t} \sin(2t) & \text{, for } 0 \le t \le \pi \\ \tfrac{1}{2} \left(1 + e^{\pi}\right) e^{-t} \sin(2t) & \text{, for } \pi \le t \le 2\pi \\ \tfrac{1}{2} \left(1 + e^{\pi} + e^{2\pi}\right) e^{-t} \sin(2t) & \text{, for } 2\pi \le t \le 3\pi \\ \vdots & \end{cases} \end{aligned}$$

For $n\pi < t < (n+1)\pi$ we have $y(t) = \frac{1}{2}\left(1 + e^{\pi} + e^{2\pi} + \cdots + e^{n\pi}\right) e^{-t} \sin(2t) = \dfrac{e^{(n+1)\pi} - 1}{e^{\pi} - 1} \cdot \frac{1}{2} e^{-t} \sin(2t)$ (where we used the formula for the sum of a geometric series), and so

$$y'(t) = \frac{e^{(n+1)\pi} - 1}{e^{\pi} - 1} \Big(-\tfrac{1}{2} e^{-t} \sin(2t) + e^{-t} \cos(2t) \Big) .$$

The velocity of the object just after the n^{th} time it is struck is $\lim\limits_{t \to n\pi^+} y'(t) = \dfrac{e^{(n+1)\pi} - 1}{e^{\pi} - 1} \cdot e^{-n\pi} = \dfrac{e^{\pi} - e^{-n\pi}}{e^{\pi} - 1}$.

Asymptotically, this velocity approaches $\lim\limits_{n \to \infty} \dfrac{e^{\pi} - e^{-n\pi}}{e^{\pi} - 1} = \dfrac{e^{\pi}}{e^{\pi} - 1}$.

6: (a) Find the inverse Laplace transform of each of the following functions $F(s)$.

(i) $F(s) = \dfrac{8}{(s^2+2s+2)^3}$. (ii) $F(s) = \dfrac{8s}{(s^2+2s+2)^3}$.

Solution: For part (i), let $F(s) = \dfrac{8}{(s^2+2s+2)^3}$. Note that $F(s) = G(s+1)^3$ where $G(s) = \dfrac{2}{s^2+1}$. Also, let $g(t) = \mathcal{L}^{-1}\big[F(s)\big](t) = 2\sin t$. Then (as we showed in class) we have

$$\begin{aligned}(g*g)(t) &= \int_{u=0}^{t} 4\sin(t-u)\sin(u)\,du = \int_{u=0}^{t} 2\big(\cos(t-2u)-\cos t\big)du \\ &= \Big[-\sin(t-2u)-(\cos t)u\Big]_{u=0}^{t} = -\sin(-t)-2t\cos t+\sin t \\ &= 2\sin t - 2t\cos t\,,\end{aligned}$$

and so

$$\begin{aligned}\big(g*(g*g)\big)(t) &= \int_{u=0}^{t} 4\sin(t-u)(\sin u - u\cos u)\,du \\ &= \int_{u=0}^{t} 4\sin(t-u)\sin u - 4u\sin(t-u)\cos u\;du \\ &= \int_{u=0}^{t} 2\cos(t-2u) - 2\cos t - 2u\sin t - 2u\sin(t-2u)\;du \\ &= \Big[-\sin(t-2u) - 2u\cos t - u^2\sin t - u\cos(t-2u) - \tfrac{1}{2}\sin(t-2u)\Big]_{u=0}^{t} \\ &= -\sin(-t) - 2t\cos t - t^2\sin t - t\cos(-t) - \tfrac{1}{2}\sin(-t) + \sin t + \tfrac{1}{2}\sin t \\ &= 3\sin t - 3t\cos t - t^2\sin t\,.\end{aligned}$$

Note that $\mathcal{L}\big[(g*g*g)(t)\big](s) = \mathcal{L}[g(t)]^3(s) = G(s)^3$, and so $\mathcal{L}\big[e^{-t}(g*g*g)(t)\big](s) = G(s+1)^3 = F(s)$. Thus $\mathcal{L}^{-1}\big[F(s)\big](t) = e^{-t}\big(3\sin t - 3t\cos t - t^2\sin t\big)$.

For part (ii), recall that $\mathcal{L}^{-1}\left[\frac{-2}{(s^2+1)^2}\right] = -\sin t + t\cos t$ (we showed this in class, and it also follows from the work we did in part (i)). We have

$$\frac{8s}{(s^2+1)^3} = \frac{d}{ds}\left(\frac{-2}{(s^2+1)^2}\right) = \frac{d}{ds}\mathcal{L}\big[-\sin t + t\cos t\big] = \mathcal{L}\big[t\sin t - t^2\cos t\big]$$

and so

$$\mathcal{L}^{-1}\left[\frac{8(s+1)}{(s^2+2s+2)^3}\right] = e^{-t}(t\sin t - t^2\cos t)\,.$$

Together with the result from part (i) this gives

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{8s}{(s^2+2s+2)^3}\right] &= \mathcal{L}^{-1}\left[\frac{8(s+1)}{(s^2+2s+2)^3} - \frac{8}{(s^2+2s+2)^3}\right] \\ &= e^{-t}\big((t\sin t - t^2\cos t) - (3\sin t - 3t\cos t - t^2\sin t)\big) \\ &= e^{-t}\big((-3+t+t^2)\sin t + (3t-t^2)\cos t\big)\,.\end{aligned}$$

(b) Find a formula for the solution to the IVP $y'' + 2y' + 5y = g(x)$ with $y(0) = 2$, $y'(0) = 4$.

Solution: Take the Laplace transform of both sides to get

$$(-4 - 2s + s^2Y) + 2(-2 + sY) + 5Y = G$$
$$(s^2 + 2s + 5)Y = G + 2s + 8$$
$$Y = \frac{G}{s^2 + 2s + 5} + \frac{2s + 8}{s^2 + 2s + 5} = \frac{G}{(s + 1)^2 + 4} + \frac{2(s + 1) + 6}{(s + 1)^2 + 4}$$

where $G(s) = \mathcal{L}\big[g(t)\big](s)$. Take the inverse Laplace transform to get

$$\begin{aligned} y &= \tfrac{1}{2}e^{-t}\sin(2t) * g(t) + 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t) \\ &= \int_{u=0}^{t} \tfrac{1}{2}e^{-(t-u)}\sin\big(2(t - u)\big)g(u)\,du + 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)\,. \end{aligned}$$