MATH 218 Differential Equations, Solutions to Assignment 8

1: Find the Laplace transform of each of the following functions $f(t)$.

(a) $f(t) = t^3 e^{-t/2}$.

Solution: $\mathcal{L}\big[t^3 e^{-t/2}\big] = \dfrac{6}{\left(s+\frac{1}{2}\right)^4}$.

(b) $(t+e^t)^3$.

Solution: $f(t) = t^3 + 3t^2 e^t + 3te^{2t} + e^{3t}$, so $\mathcal{L}\big[f(t)\big] = \dfrac{6}{s^4} + \dfrac{6}{(s-1)^3} + \dfrac{3}{(s-2)^2} + \dfrac{1}{(s-3)}$

(c) $f(t) = a\sinh(\omega t) + b\cosh(\omega t)$.

Solution: Note that $\sinh(\omega t) = \frac{1}{2}\big(e^{\omega t} - e^{-\omega t}\big)$ so $\mathcal{L}\big[\sinh(\omega t)\big] = \frac{1}{2}\left(\dfrac{1}{s-\omega} - \dfrac{1}{s+\omega}\right) = \dfrac{\omega}{s^2-\omega^2}$, and $\cosh(\omega t) = \frac{1}{2}\big(e^{\omega t} + e^{-\omega t}\big)$ so $\mathcal{L}\big[\cosh(\omega t)\big] = \frac{1}{2}\left(\dfrac{1}{s-\omega} + \dfrac{1}{s+\omega}\right) = \dfrac{s}{s^2-\omega^2}$. Thus

$$\mathcal{L}\big[a\sinh(\omega t) + b\cosh(\omega t)\big] = \frac{a\omega + bs}{s^2-\omega^2}\,.$$

2: Find the Laplace transform of each of the following functions $f(t)$.

(a) $f(t) = \begin{cases} t & \text{, if } 0 \le t < 1, \\ 3-2t & \text{, if } 1 \le t < 2, \\ t-3 & \text{, if } 2 \le t < 3, \\ 0 & \text{, if } 3 \le t\,. \end{cases}$

Solution: Note that $f(t) = t\,H(t) - 3(t-1)H(t-1) + 3(t-2)H(t-2) - (t-3)H(t-3)$ and so

$$\mathcal{L}\big[f(t)\big] = \frac{1}{s^2} - \frac{3e^{-s}}{s^2} + \frac{3e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s^2} = \frac{(1-e^{-s})^3}{s^2}\,.$$

(b) $f(t) = 2e^t\cos\left(2t - \frac{\pi}{3}\right)$.

Solution: Note that $f(t) = e^t\left(\cos 2t + \sqrt{3}\sin 2t\right)$, so $\mathcal{L}\big[f(t)\big] = \dfrac{s-1}{(s-1)^2+4} + \dfrac{\sqrt{3}\cdot 2}{(s-1)^2+4} = \dfrac{s-1+2\sqrt{3}}{s^2-2s+5}$.

(c) $f(t) = t\,e^{-t}\sin 3t$.

Solution: $\mathcal{L}\big[t\,e^{-t}\sin 3t\big] = -\dfrac{d}{ds}\mathcal{L}\big[e^{-t}\sin 3t\big] = -\dfrac{d}{ds}\left(\dfrac{3}{(s+1)^2+9}\right) = \dfrac{3\cdot 2(s+1)}{\big((s+1)^2+9\big)^2} = \dfrac{6(s+1)}{(s^2+2s+10)^2}\,.$

3: Find the inverse Laplace transform of the following functions $F(s)$.

(a) $F(s) = \dfrac{s-3}{s^3+s^2+s+1}$.

Solution: Note that $s^3+s^2+s+1 = (s+1)(s^2+1)$. To get $\dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1} = \dfrac{s-3}{(s+1)(s^2+1)}$ we need $A(s^2+1)+(Bs+C)(s+1) = s-3$. Equate coefficients and also set $s=-1$ to get the equations $A+B=0$, $B+C=1$, $A+C=-3$ and $2A=-4$. Solve these to get $A=-2$, $B=2$ and $C=-1$. Thus we have

$$\mathcal{L}^{-1}\left[\frac{s-3}{s^3+s^2+s+1}\right] = \mathcal{L}^{-1}\left[\frac{-2}{s+1} + \frac{2s-1}{s^2+1}\right] = -2e^{-t} + 2\cos t - \sin t\,.$$

(b) $F(s) = \dfrac{5s+7}{s^3+4s^2+s-6}$.

Solution: Note that $s^3+4s^2+s-6 = (s-1)(s^2+5s+6) = (s-1)(s+2)(s+3)$. We have

$$\mathcal{L}^{-1}\left[\frac{5s+7}{s^3+4s^2+s-6}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-1} + \frac{1}{s+2} - \frac{2}{s+3}\right] = e^t + e^{-2t} - 2e^{-3t}\,.$$

(c) $F(s) = \dfrac{s-2}{s^4+4s^3+8s^2}$.

Solution: Note that $s^4+4s^3+8s^2 = s^2(s^2+4s+8)$. To get $\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4s+8} = \dfrac{s-2}{s^2(s^2+4s+8)}$ we need $As(s^2+4s+8)+B(s^2+4s+8)+(Cs+D)s^2 = s-2$. Equate coefficients to get $A+C=0$, $4A+B+D=0$, $8A=4B=1$ and $8B=-2$. Solve these to get $B=-\frac{1}{4}$, $A=\frac{1}{4}$, $C=-\frac{1}{4}$ and $D=-\frac{3}{4}$. Thus

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s-2}{s^4+4s^3+8s^2}\right] &= \mathcal{L}^{-1}\left[\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}s+\frac{3}{4}}{(s+2)^2+4}\right] = \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^2} - \frac{(s+2)+1}{(s+2)^2+4}\right] \\ &= \tfrac{1}{4}\Big(1 - t - e^{-2t}\big(\cos 2t + \tfrac{1}{2}\sin 2t\big)\Big)\,.\end{aligned}$$

4: Find the inverse Laplace transform of the following functions $F(s)$.

(a) $F(s) = \dfrac{(s+1)e^{-s}}{s(s+2)}$.

Solution: Let $f(t) = \mathcal{L}^{-1}\left[\dfrac{s+1}{s(s+2)}\right] = \mathcal{L}^{-1}\left[\dfrac{\frac{1}{2}}{s} + \dfrac{\frac{1}{2}}{s+2}\right] = \frac{1}{2} + \frac{1}{2}\,e^{-2t} = \frac{1}{2}\big(1+e^{-2t}\big)$. Then we have

$$\mathcal{L}^{-1}\left[\frac{(s+1)e^{-s}}{s(s+2)}\right] = f(t-1)H(t-1) = \tfrac{1}{2}\left(1+e^{-2(t-1)}\right)H(t-1)\,.$$

(b) $F(s) = \dfrac{4+se^{-2s}}{s^2-4}$.

Solution: Note that we have $\mathcal{L}^{-1}\left[\dfrac{4}{s^2-4}\right] = \mathcal{L}^{-1}\left[\dfrac{1}{s-2} - \dfrac{1}{s+2}\right] = e^{2t} - e^{-2t} = 2\sinh(2t)$ and we have $\mathcal{L}^{-1}\left[\dfrac{s}{s^2-4}\right] = \mathcal{L}^{-1}\left[\dfrac{\frac{1}{2}}{s-2} + \dfrac{\frac{1}{2}}{s+2}\right] = \frac{1}{2}\big(e^{2t}+e^{-2t}\big) = \cosh(2t)$, and so

$$\mathcal{L}^{-1}\left[\frac{4+se^{-2s}}{s^2-4}\right] = 2\sinh(2t) + \cosh\big(2(t-2)\big)H(t-2)\,.$$

(c) $F(s) = \dfrac{s^2-1}{(s^2+1)^2}$.

Solution: Note first that, using the substitution $\tan\theta = x$, $\sec^2\theta\,d\theta = dx$ we have

$$\begin{aligned}\frac{s^2-1}{(s^2+1)^2}\,ds &= \int \frac{\tan^2\theta - 1}{\sec^2\theta}\,\sec\theta\,d\theta = \int \frac{\sec^2\theta - 2}{\sec^2\theta}\,d\theta = \int 1 - 2\cos^2\theta\,d\theta \\ &= \int -\cos 2\theta\,d\theta = -\tfrac{1}{2}\sin 2\theta = -\sin\theta\cos\theta = \frac{-s}{s^2+1}\,.\end{aligned}$$

so we have $\dfrac{s^2-1}{(s^2+1)^2} = \dfrac{d}{ds}\left(\dfrac{-s}{s^2+1}\right) = \dfrac{d}{ds}\mathcal{L}\big[-\cos t\big] = \mathcal{L}\big[t\cos t\big]$. Thus $\mathcal{L}^{-1}\left[\dfrac{s^2-1}{(s^2+1)^2}\right] = t\cos t$.

5: (a) Show that if $f(t)$ is periodic with period T (this means that $f(t+T)=f(t)$ for all t) then

$$\mathcal{L}\big[f(t)\big](s)=\frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt\,.$$

Solution: Let $f(t)$ be periodic with period T. Then, using the substitution $u=t-T$ and then later changing u back into t, we have

$$\begin{aligned}\mathcal{L}\big[f(t)\big](s)&=\int_0^\infty f(t)e^{-st}\,dt=\int_0^T f(t)e^{-st}\,dt+\int_T^\infty f(t)e^{-st}\,dt\\&=\int_0^T f(t)e^{-st}\,dt+\int_{u=0}^\infty f(u+T)e^{-s(u+T)}\,du\\&=\int_0^T f(t)e^{-st}\,dt+e^{-Ts}\int_{u=0}^\infty f(u)e^{-su}\,du\\&=\int_0^T f(t)e^{-st}\,dt+e^{-Ts}\int_0^\infty f(t)e^{-st}\,dt\\&=\int_0^T f(t)e^{-st}\,dt+e^{-Ts}\mathcal{L}\big[f(t)\big](s)\end{aligned}$$

and so $\mathcal{L}\big[f(t)\big](s)\big(1-e^{-Ts}\big)=\int_0^T f(t)e^{-st}\,dt$, that is $\mathcal{L}\big[f(t)\big](s)=\dfrac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$.

(b) Find the Laplace transform of the sawtooth function $f(t)$ which is given by $f(t)=t-n$ for $n\le t<n+1$ for each integer $n\ge 0$.

Solution: We provide two solutions. The first solution makes use of the formula found in part (a). Since $f(t)$ has period $T=1$, the above formula gives $\mathcal{L}\big[f(t)\big](s)=\dfrac{1}{1-e^{-s}}\int_0^1 t\,e^{-st}\,dt$. Integrate by parts using $u=t$, $du=dt$, $v=-\frac{1}{s}e^{-st}$ and $dv=e^{-st}\,dt$ to get

$$\begin{aligned}\mathcal{L}\big[f(t)\big](s)&=\frac{1}{1-e^{-s}}\left[-\frac{t}{s}\,e^{-st}+\int\frac{1}{s}\,e^{-st}\,dt\right]_0^1=\frac{1}{1-e^{-s}}\left[-\frac{t}{s}\,e^{-st}-\frac{1}{s^2}\,e^{-st}\right]_0^1\\&=\frac{1}{1-e^{-s}}\left(-\frac{1}{s}\,e^{-s}-\frac{1}{s^2}\,e^{-1}+\frac{1}{s^2}\right)=\frac{1-(s+1)e^{-s}}{s^2\,(1-e^{-s})}\,.\end{aligned}$$

For our second solution, we note that $f(t)=tH(t)-H(t-1)-H(t-2)-\cdots=tH(t)-\sum_{n=1}^\infty H(t-n)$, so using the formula for the sum of a geometric series, we have

$$\mathcal{L}\big[f(t)\big](s)=\frac{1}{s^2}-\sum_{n=1}^\infty\frac{e^{-ns}}{s}=\frac{1}{s^2}-\frac{1}{s}\sum_{n=1}^\infty\big(e^{-s}\big)^n=\frac{1}{s^2}-\frac{1}{s}\,\frac{e^{-s}}{1-e^{-s}}\,.$$