MATH 218 Differential Equations, Solutions to Assignment 7

1: Find the 5^{th} Taylor polynomial centered at 0 for the solution to the IVP $(1-x)\,y''-2y=4$ with $y(0)=1$ and $y'(0)=3$.

Solution: We try a solution of the form $y=\sum_{n=0}^{\infty} a_n x^n$. Then $y'=\sum_{n=1}^{\infty} n\,a_n x^{n-1}$ and $y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Put these in the DE to get

$$\begin{aligned}
0&=(1-x)y''-2y-4\\
&=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}-\sum_{n=0}^{\infty} 2a_n x^n-4\\
&=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=1}^{\infty}(m+1)m\,a_{m+1}x^m-\sum_{m=0}^{\infty}2a_m x^m-4\\
&=(2a_2-2a_0-4)x^0+\sum_{m=1}^{\infty}\big((m+2)(m+1)a_{m+2}-m(m+1)a_{m+1}-2a_m\big)x^m\,.
\end{aligned}$$

The coefficients must all vanish so we have $a_2=a_0+2$ and for $m\geq 1$ we have $a_{m+2}=\dfrac{m(m+1)a_{m+1}+2a_m}{(m+2)(m+1)}$. To get $y(0)=1$ we need $a_0=1$ and to get $y'(0)=3$ we need $a_1=3$. The recursion formula then gives $a_2=a_0+2=3$, $a_3=\frac{1\cdot 2\,a_3+2a_2}{2\cdot 3}=\frac{6+6}{6}=2$, $a_4=\frac{2\cdot 3\,a_3+2a_2}{3\cdot 4}=\frac{12+6}{12}=\frac{3}{2}$, and $a_5=\frac{3\cdot 4\,a_4+2a_3}{4\cdot 5}=\frac{18+4}{20}=\frac{11}{10}$. Thus the 5^{th} Taylor polynomial is $T_5(x)=1+3x+3x^2+2x^3+\frac{3}{2}x^4+\frac{11}{10}x^5$.

2: Find the 5^{th} Taylor polynomial centered at 0 for the solution to the IVP $y''+2y'+e^x y=\sin x$ with $y(0)=2$ and $y'(0)=1$.

Solution: We try $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$. Then $y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots$, $y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots$, and

$$\begin{aligned}
e^x y&=\big(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\cdots\big)\big(a_0+a_1x+a_2x^2+a_3x^3+\cdots\big)\\
&=\big(a_0+(a_1+a_0)x+(a_2+a_1+\tfrac{1}{2}a_0)x^2+(a_3+a_2+\tfrac{1}{2}a_1+\tfrac{1}{6}a_0)x^3+\cdots\big)\,.
\end{aligned}$$

Put these in the DE to get

$$\begin{aligned}
&\big(2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots\big)+2\big(a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots\big)\\
&+\big(a_0+(a_1+a_0)x+(a_2+a_1+\tfrac{1}{2}a_0)x^2+(a_3+a_2+\tfrac{1}{2}a_1+\tfrac{1}{6}a_0)x^3+\cdots\big)\\
&=\big(x-\tfrac{1}{6}x^3+\cdots\big)\,.
\end{aligned}$$

Equate coefficients to get $2a_2+2a_1+a_0=0$ (1), $6a_3+4a_2+a_1+a_0=1$ (2), $12a_4+6a_3+a_2+a_1+\frac{1}{2}a_0=0$ (3) and $20a_5+8a_4+a_3+a_2+\frac{1}{2}a_1+\frac{1}{6}a_0=-\frac{1}{6}$ (4). To get $y(0)=2$ we need $a_0=2$ and to get $y'(0)=1$ we need $a_1=1$. Put these in the recursion formulas (1)-(4) to get $a_2=\frac{-2a_1-a_0}{2}=\frac{-2-2}{2}=-2$, $a_3=\frac{1-4a_2-a_1-a_0}{6}=\frac{1+8-1-2}{6}=1$, $a_4=\frac{-6a_3-a_2-a_1-\frac{1}{2}a_0}{12}=\frac{-6+2-1-1}{12}=-\frac{1}{2}$ and $a_5=\frac{-\frac{1}{6}-8a_4-a_3-a_2-\frac{1}{2}a_1-\frac{1}{6}a_0}{20}=\frac{-\frac{1}{6}+4-1+2-\frac{1}{2}-\frac{1}{3}}{20}=\frac{1}{5}$. Thus the 5^{th} Taylor polynomial is $T_5(x)=2+x-2x^2+x^3-\frac{1}{2}x^4+\frac{1}{5}x^5$.

3: Use the Power Series Method to solve the DE $y'' + (x-1)y' + y = 0$. Find two linearly independent power series solutions, one satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$, and the other satisfying $y(0) = 0$, $y'(0) = 1$. For each solution, state the recurrence relation for the coefficients, and find the 5^{th} Taylor polynomial centered at 0.

Solution: We try $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Put these in the DE to get

$$\begin{aligned}
0 &= y'' + (x-1)y' + y \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=1}^{\infty} m\, a_m x^m - \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^m \\
&= (2a_2 - a_1 + a_0)x^0 + \sum_{m=1}^{\infty} \big((m+2)(m+1)a_{m+2} - (m+1)a_{m+1} + (m+1)a_m\big)x^m .
\end{aligned}$$

All coefficients vanish, so $a_2 = \dfrac{a_1 - a_0}{2}$ and $a_{m+2} = \dfrac{(m+1)a_{m+1} - (m+1)a_m}{(m+2)(m+1)} = \dfrac{a_{m+1} - a_m}{m+2}$ for $m \geq 1$. If $a_0 = 1$ and $a_1 = 0$ then the recursion formulas give $a_2 = \frac{0-1}{2} = -\frac{1}{2}$, $a_3 = \frac{-\frac{1}{2}-0}{3} = -\frac{1}{6}$, $a_4 = \frac{-\frac{1}{6}+\frac{1}{2}}{4} = \frac{1}{12}$ and $a_5 = \frac{\frac{1}{12}+\frac{1}{6}}{5} = \frac{1}{20}$, so the 5^{th} Taylor polynomial is

$$T_5(y_1) = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \tfrac{1}{20}x^5 .$$

If $a_0 = 0$ and $a_1 = 1$ then the recursion formulas give $a_2 = \frac{1-0}{2} = \frac{1}{2}$, $a_3 = \frac{\frac{1}{2}-1}{3} = -\frac{1}{6}$, $a_4 = \frac{-\frac{1}{6}-\frac{1}{2}}{4} = -\frac{1}{6}$ and $a_5 = \frac{-\frac{1}{6}+\frac{1}{6}}{5} = 0$ and so the 5^{th} Taylor polynomial for the solution y_2 is

$$T_5(y_2) = x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 .$$

4: Use Frobenius' Method to solve the DE $4xy'' + 2y' = y$. Find two linearly independent series solutions. For each solution, solve the recurrence relation to obtain an explicit formula for the n^{th} coefficient, then find a closed form formula for the solution.

Solution: We try $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ so $y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)x^{n+r-2}$. Put these in the DE to get

$$\begin{aligned}
0 &= 4xy'' + 2y' - y \\
&= \sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= x^r \left(\sum_{m=-1}^{\infty} 4(m+r+1)(m+r)a_{m+1}x^m + \sum_{m=-1}^{\infty} 2(m+r+1)a_{m+1}x^m - \sum_{m=0}^{\infty} a_m x^m \right) \\
&= x^r \left(\sum_{m=-1}^{\infty} 2(m+r+1)(2m+2r+1)a_{m+1}x^m - \sum_{m=0}^{\infty} a_m x^m \right) \\
&= x^r \left(2r(2r-1)a_0 x^{-1} + \sum_{m=0}^{\infty} \big(2(m+r+1)(2m+2r+1)a_{m+1} - a_m\big)x^m \right) .
\end{aligned}$$

All coefficients must vanish, so we have $r(2r-1) = 0$ and $a_{m+1} = \dfrac{a_m}{2(m+r+1)(2m+2r+1)}$ for ≥ 0. When $r = 0$, the recursion formula becomes $a_{m+1} = \dfrac{a_m}{2(m+1)(2m+1)} = \dfrac{a_m}{(2m+1)(2m+2)}$, so if $a_0 = 1$ then we get $a_1 = \frac{1}{1\cdot 2}$, $a_2 = \frac{1}{1\cdot 2\cdot 3\cdot 4}$, $a_3 = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$, and in general $a_n = \frac{1}{(2n)!}$. In this case the solution is

$$y_1 = x^0 \left(\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} \right) = \cosh \sqrt{x} \, .$$

When $r = \frac{1}{2}$ the recursion formula becomes $a_{m+1} = \dfrac{a_m}{2\left(m+\frac{3}{2}\right)(2m+2)} = \dfrac{a_m}{(2m+2)(2m+3)}$, so if $a_0 = 1$ then we get $a_1 = \frac{1}{2\cdot 3}$, $a_2 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$ and in general $a_n = \frac{1}{(2n+1)!}$. In this case the solution is

$$y_2 = x^{1/2} \left(\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!} \right) = \sqrt{x} \cdot \frac{\sinh \sqrt{x}}{\sqrt{x}} = \sinh \sqrt{x} \, .$$

The general solution is $y = a \sinh \sqrt{x} + b \cosh \sqrt{x}$.

5: Use Frobenius' Method to solve the DE $3x^2y'' + x(x-1)y' + y = 0$. Find two linearly independent series solutions. For each solution, solve the recurrence relation to obtain an explicit formula for the n^{th} coefficient. Find a closed form formula for one of the two solutions.

Solution: We try $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ so $y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$. Put these in the DE to get

$$\begin{aligned}
0 &= 3x^2y'' + x(x-1)y' + y \\
&= \sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= x^r\left(\sum_{m=0}^{\infty} 3(m+r)(m+r-1)a_m x^m + \sum_{m=1}^{\infty}(m+r-1)a_{m-1}x^m - \sum_{m=0}^{\infty}(m+r)a_m x^m + \sum_{m=0}^{\infty} a_m x^m\right) \\
&= x^r\left(\sum_{m=0}^{\infty}\big(3(m+r)(m+r-1) - (m+r) + 1\big)a_m x^m + \sum_{m=1}^{\infty}(m+r-1)a_{m-1}x^m\right) \\
&= x^r\left(\big(3r(r-1) - r + 1\big)a_0 x^0 + \sum_{m=1}^{\infty}\Big(\big(3(m+r)(m+r-1) - (m+r) + 1\big)a_m + (m+r-1)a_{m-1}\Big)x^m\right).
\end{aligned}$$

All coefficients must vanish, so we have $3r(r-1) - r + 1 = 0$, that is $3r^2 - 4r + 1 = 0$ or equivalently $(3r-1)(r-1) = 0$ so $r = 1$ or $r = \frac{1}{3}$, and we have $a_m = \dfrac{-(m+r-1)a_{m-1}}{3(m+r)(m+r-1) - (m+r) + 1}$. When $r = 1$ the recursion formula becomes $a_m = \dfrac{-m\,a_{m-1}}{3(m+1)(m) - m} = \dfrac{-a_{m-1}}{3m+2}$. If we take $a_0 = 1$ then we have $a_1 = -\frac{1}{5}$, $a_2 = \frac{1}{5\cdot 8}$, $a_3 = -\frac{1}{5\cdot 8\cdot 11}$, and in general $a_n = \frac{(-1)^n}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$. In this case we obtain the solution

$$y_1 = x^1\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}\right) = x - \tfrac{1}{5}x^2 + \tfrac{1}{5\cdot 8}x^3 - \tfrac{1}{5\cdot 8\cdot 11}x^4 + \cdots.$$

When $r = \frac{1}{3}$ the recursion formula becomes

$$a_m = \frac{-\left(m - \frac{2}{3}\right)a_{m-1}}{3\left(m + \frac{1}{3}\right)\left(m - \frac{2}{3}\right) - m + \frac{2}{3}} = \frac{-(3m-2)a_{m-1}}{(3m+1)(3m-2) - 3m + 2} = -\frac{a_{m-1}}{3m}.$$

If we set $a_0 = 1$ then we obtain $a_2 = -\frac{1}{3}$, $a_2 = \frac{1}{3\cdot 6}$, $a_3 = -\frac{1}{3\cdot 6\cdot 9}$, and in general $a_n = \frac{(-1)^n}{3\cdot 6\cdot 9\cdot\cdots\cdot(3n)} = \frac{(-1)^n}{3^n n!}$. In this case we obtain the solution

$$y_2 = x^{1/3}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}\right) = x^{1/3}e^{-x/3}.$$