MATH 218 Differential Equations, Solutions to Assignment 6

1: An object of mass $m = 2$ kg is attached to a spring with spring-constant $k = 18$ N/m. Let $x(t)$ be the object's displacement, in meters, from the equilibrium position at time t seconds. Suppose that $x(0) = 2$ and $x'(0) = 6$. Ignoring air resistance, find the smallest positive value of t such that $x(t) = 0$.

Solution: The displacement $x(t)$ satisfies the DE $mx'' + kx = 0$, that is $2x'' + 18x = 0$ or equivalently $x'' + 9x = 0$. The characteristic equation is $r^2 + 9 = 0$ and the roots are $r = \pm 3i$. The solution to the DE is $x = A\sin(3t) + B\cos(3t)$ and then $x' = 3A\cos(3t) - 3B\sin(3t)$. To get $x(0) = 2$ we need $B = 2$, and to get $x'(0) = 6$ we need $3A = 6$ so $A = 2$. Thus the displacement is

$$x(t) = 2\sin(3t) + 2\cos(3t)\,.$$

We have $x(t) = 0 \iff 2\sin(2t) + 2\cos(3t) = 0 \iff \sin(2t) = -\cos(3t) \iff \tan(3t) = -1 \iff 3t = -\frac{\pi}{4} + \pi\, k \iff t = -\frac{\pi}{12} + \frac{\pi}{3}\, k$ for some integer k. Thus the first positive value of t for which $x(t) = 0$ is $t = -\frac{\pi}{12} + \frac{\pi}{3} = \frac{\pi}{4}$.

2: An object of mass $m = 2$ kg is attached to a spring with spring-constant $k = 8$ N/m in a liquid where the damping-constant is $c = 10$ kg/s. Let $x(t)$ be the displacement, in meters, from the equilibrium position at time t seconds.

(a) Determine whether the system is overdamped, underdamped, or critically damped.

Solution: The displacement satisfies the DE $mx'' + cx' + kx = 0$, that is $2x'' + 10x' + 8x = 0$, or equivalently $x'' + 5x' + 4x = 0$. The characteristic equation is $r^2 + 5r + 4 = 0$, that is $(r+1)(r+4) = 0$. Since there are two real solutions to the characteristic equation, the system is overdamped.

(b) Given that $x(0) = 1$ and $x'(0) = 5$, find the time t at which $x(t)$ reaches its maximum.

Solution: Since tho roots of the characteristic equation are $r = -1, -4$, the solution to the above DE is $x = Ae^{-t} + Be^{-4t}$, and then $x' = -Ae^{-t} - 4Be^{-4t}$. To get $x(0) = 1$ we need $A + B = 1$ (1), and to get $x'(0) = 5$ we need $-A - 4B = 5$ (2). Solve equations (1) and (2) to get $A = 3$ and $B = -2$, so the displacement is given by

$$x = 3e^{-t} - 2e^{-4t}\,.$$

To find the maximum value of $x(t)$, we set $x'(t) = 0$. We have $x'(t) = 0 \iff -3e^{-t} + 8e^{-4t} = 0 \iff -3 + 8e^{-3t} = 0 \iff e^{-3t} = \frac{3}{8} \iff -3t = \ln\frac{3}{8} = -\ln\frac{8}{3} \iff t = \frac{1}{3}\ln\frac{8}{3}$. It is clear from the physical nature of the problem that this value of t gives the maximum value for $x(t)$ (not the minimum), but we could verify this mathematically in several ways (for example by using the first or second derivative test).

3: An LRC series circuit consists of an inductor of inductance $L = 1$ henries, a resistor of resistance $R = 2$ ohms, and a capacitor of capacitance $C = \frac{1}{5}$ farads, and there is an impressed voltage of $E = \sin 2t$ volts. Let $q = q(t)$ coulombs be the charge on the capacitor at time t seconds.

(a) Find the asymptotic (that is limiting or eventual) amplitude and phase of $q(t)$.

Solution: The charge on the capacitor satisfies the DE $Lq'' + Rq' + \frac{1}{C}q = E$, that is $x'' + 2x' + 5x = \sin(2t)$. The characteristic equation is $r^2 + 2r + 5 = 0$, and the roots are $r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$, so the solution to the homogeneous DE is $q = Aq_1 + Bq_2$ where $q_1 = e^{-t}\sin(2t)$ and $q_2 = e^{-t}\cos(2t)$. To find a particular solution to the non-homogeneous DE, we try $q = q_p = a\sin(2t) + b\cos(2t)$. Then $q' = 2a\cos(2t) - 2b\sin(2t)$ and $q'' = -4a\sin(2t) - 4b\cos(2t)$. Put these in the DE to get

$$\begin{aligned}\sin(2t) &= -4a\sin(2t) - 4b\cos(2t) + 4a\cos(2t) - 4b\sin(2t) + 5a\sin(2t) + 5b\cos(2t)\\ &= (a - 4b)\sin(2t) + (4a + b)\cos(2t)\,.\end{aligned}$$

Since $\sin(2t)$ and $\cos(2t)$ are linearly independent, we can equate coefficients to get $a - 4b = 1$ (1) and $4a + b = 0$ (2). Solve these two equations to get $a = \frac{1}{17}$ and $b = -\frac{4}{17}$. Thus we obtain the particular solution $q_p = \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t)$, and so the general solution to the non-homogeneous DE is

$$q(t) = Ae^{-x}\sin(2x) + Be^{-x}\cos(2x) + \tfrac{1}{17}\sin(2t) - \tfrac{4}{17}\cos(2t)\,.$$

Since $\lim\limits_{t\to 0} Ae^{-x}\sin(2x) + Be^{-x}\cos(2x) = 0$, we see that $q(t)$ is asymptotic to the particular solution $q_p(t)$. We have $q(t) \sim q_p = \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t) = M\sin(2t + \phi)$ where $M = \sqrt{\left(\frac{1}{17}\right)^2 + \left(\frac{4}{17}\right)^2} = \frac{1}{\sqrt{17}}$, and ϕ is the angle given by $\sin\phi = \frac{-4/17}{1/\sqrt{17}} = \frac{-4}{\sqrt{17}}$ and $\cos\phi = \frac{1/17}{\sqrt{17}} = \frac{1}{\sqrt{17}}$, that is $\phi = -\tan^{-1} 4$. These quantities, $M = \frac{1}{\sqrt{17}}$ and $\phi = \tan^{-1} 4$, are the asymptotic amplitude and phase of $q(t)$.

(b) Given that $q(0) = 0$ and $q'(0) = 0$, find a formula for $q(t)$.

Solution: In part (a) we found that $q(t) = Ae^{-x}\sin(2x) + Be^{-x}\cos(2x) + \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t)$, and so we have $q'(t) = -Ae^{-t}\sin(2t) + 2Ae^{-t}\cos(2t) - Be^{-t}\cos(2t) - 2Be^{-t}\sin(2t) + \frac{2}{17}\cos(2t) - \frac{8}{17}\sin(2t)$. To get $q(0) = 0$ we need $B - \frac{4}{17} = 0$ (1), and to get $q'(0) = 0$ we need $2A - B + \frac{2}{17} = 0$ (2). Solve equations (1) and (2) to get $A = \frac{1}{17}$ and $B = \frac{4}{17}$. Thus the charge on the capacitor is given by

$$q(t) = \tfrac{1}{17}e^{-t}\sin(2t) + \tfrac{4}{17}e^{-t}\cos(2t) + \tfrac{1}{17}\sin(2t) - \tfrac{4}{17}\cos(2t)\,.$$

4: On a planet where the surface gravitational constant happens to be $g = 10$ m/s^2, an object of mass $m = 1$ kg is thrown upwards from the ground at 20 m/s. Let $x(t)$ be its height in meters at time t seconds. Suppose the force at time t due to air resistance is $-\frac{1}{10}\,x'(t)$ newtons.

(a) Find the time t at which the object reaches its maximum height.

Solution: The total force acting on the object is $F = -mg - \frac{1}{10}x' = -10 - \frac{1}{10}x'$, so Newton's Second Law gives $mx'' = -10 - \frac{1}{10}x'$, that is $x'' + \frac{1}{10}x' = -10$. This is a linear DE for $v = x'$ since we can write it as $v' + \frac{1}{10}v = -10$. An integrating factor is $\lambda = e^{\int \frac{1}{10}\,dt} = e^{t/10}$, and the general solution is

$$v(t) = e^{-t/10}\int -10\,e^{t/10}\,dt = e^{-t/10}\big(-100\,e^{t/10} + c_1\big) = c_1\,e^{-t/10} - 100\,.$$

Put in $v(0) = x'(0) = 20$ to get $c_1 - 100 = 20$, so $c_1 = 120$ and we have

$$v(t) = 120\,e^{-t/10} - 100\,.$$

It reaches its maximum height when $v(t) = 0$, and we have $v(t) = 0 \Longrightarrow 120\,e^{-t/10} - 100 = 0 \Longrightarrow e^{-t/10} = \frac{100}{120} = \frac{5}{6} \Longrightarrow e^{t/10} = \frac{6}{5} \Longrightarrow \frac{1}{10}\,t = \ln\left(\frac{6}{5}\right) \Longrightarrow t = 10\ln\left(\frac{6}{5}\right)$.

(b) Find $x(t)$ and (with the help of a calculator) determine whether the object takes longer on the way up to its maximum height or on the way back down to the ground.

Solution: We have $x(t) = \displaystyle\int v(t)\,dt = \int 120\,e^{-t/10} - 100\,dt = -1200\,e^{-t/10} - 100\,t + c_2$. Put in $x(0) = 0$ to get $-1200 + c_2 = 0$, so $c_2 = 1200$ and we have

$$x(t) = -1200\,e^{-t/10} - 100\,t + 1200 = 1200\big(1 - e^{-t/10}\big) - 100\,t\,.$$

By part (a), it gets to the top at $t_1 = 10\ln\left(\frac{6}{5}\right)$. Consider its position at $t_2 = 2t_1 = 20\ln\left(\frac{6}{5}\right)$. If it takes longer on the way up, then it will land before $t = t_2$ and then $x(t_2) < 0$. If it takes longer on the way back down, then it will not yet have landed when $t = t_2$ and so we will have $x(t_2) > 0$. We have

$$x(t_2) = 1200\big(1 - e^{-2\ln(6/5)}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 1200\big(1 - \tfrac{25}{36}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 100\big(\tfrac{11}{3} - 20\ln\left(\tfrac{6}{5}\right)\big)\,.$$

A calculator shows that $20\ln\left(\frac{6}{5}\right) \cong 3.64 < \frac{11}{3}$, so $x(t_2) > 0$, and so it takes longer on the way back down.

5: An object of mass m falls towards the Earth. The force due to gravity is $F = -\dfrac{GMm}{x^2}$ where x is the distance from the center of the Earth to the object, G is the gravitational constant and M is the mass of the Earth.

(a) Given that $x(0) = x_0$ and $x'(0) = 0$, find a formula for the velocity x' in terms of x.

Solution: We have $F = -\dfrac{GMm}{x^2}$ and $F = ma = mx''$, and so $x(t)$ satisfies the DE

$$x'' = -\frac{GM}{x^2}\ , \text{ with } x(0) = 0 \text{ and } x'(0) = x_0\,.$$

The independent variable t does not occur explicitly in the DE, so we let $x' = v$ and $x'' = v\,v'$ where $v' = \frac{dv}{dx}$. The DE becomes $v\,v' = -\dfrac{GM}{x^2}$. Integrate both sides to get $\frac{1}{2}\,v^2 = \dfrac{GM}{x} + c_1$. Put $x = x_0$ and $v = 0$ to get $c_1 = -\dfrac{GM}{x_0}$, and so we have $\frac{1}{2}\,v^2 = GM\left(\frac{1}{x} - \frac{1}{x_0}\right)$, that is $v = \pm\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. We are interested in the case that $v = x' \leq 0$, so

$$v = -\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}\,.$$

We remark that (if you know some physics) this formula can also be obtained using conservation of energy.

(b) Find a formula for time t in terms of x, and then find the time at which $x = \frac{1}{2}\,x_0$.

Solution: Replace v by x' again to get $x' = -\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. This DE is separable, so we write it as $\dfrac{x'}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\sqrt{2GM}$ and integrate both sides to get $\displaystyle\int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\int \sqrt{2GM}\,dt = -\sqrt{2GM}\,t + c_2$. Let I be the integral on the left. Then

$$I = \int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = \int \frac{\sqrt{x}\,dx}{\sqrt{1 - \frac{x}{x_0}}} = \int \frac{2\,x_0\sqrt{x_0}\,u^2}{\sqrt{1-u^2}}\,du\,,$$

where $u^2 = \dfrac{x}{x_0}$ so $\sqrt{x} = \sqrt{x_0}\,u$ and $2\,x_0\,u\,du = dx$. Now let $\cos\theta = u$ so that $\sin\theta = \sqrt{1-u^2}$ and $-\sin\theta\,d\theta = dx$. Then

$$\begin{aligned} I &= -\int 2\,x_0\sqrt{x_0}\,\cos^2\theta\,d\theta = -x_0\sqrt{x_0}\big(\theta + \sin\theta\cos\theta\big) + c_3 = -x_0\sqrt{x_0}\big(\cos^{-1}u + u\sqrt{1-u^2}\big) + c_3 \\ &= -x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) + c_3\,. \end{aligned}$$

Since $I = -\sqrt{2GM}\,t + c_2$, we obtain

$$-x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) = -\sqrt{2GM}\,t + c\,.$$

Put in $t = 0$ and $x = x_0$ to get $c = 0$, and so we have

$$t = \frac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0} - \left(\frac{x}{x_0}\right)^2}\,\right)\,.$$

Finally, when $x = \frac{1}{2}\,x_0$ we have $t = \dfrac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \dfrac{x_0\sqrt{x_0}\,(\pi + 2)}{4\sqrt{2GM}}$.