MATH 218 Differential Equations, Solutions to Assignment 4

1: Use the substitution $y' = u$, $y'' = u'$, where $u = u(x)$, to solve the following IVPs.

(a) $xy'' + y' = 1$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: Make the substitution $y' = u$, $y'' = u'$. The DE becomes $xu' + u = 1$. This is linear as we can write it as $u' + \frac{1}{x}u = \frac{1}{x}$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$, and the solution is given by $u = \frac{1}{x}\int 1\,dx = \frac{1}{x}(x+a) = 1 + \frac{a}{x}$. Put in $x = 1$ and $u = y' = 3$ to get $3 = 1 + a$, so $a = 2$. Thus the solution is given by $u = 1 + \frac{2}{x}$, that is $y' = 1 + \frac{2}{x}$. Integrate to get $y = \int 1 + \frac{2}{x}\,dx = x + 2\ln x + b$. Put in $x = 1$ and $y = 2$ to get $2 = 1 + b$, so $b = 1$ and the solution to the given IVP is $y = 1 + x + 2\ln x$.

(b) $y'' + x(y')^2 = 0$ with $y(0) = 2$ and $y'(0) = \frac{1}{2}$.

Solution: Make the substitution $y' = u$, $y'' = u'$. The DE becomes $u' + xu^2 = 0$. This is separable since we can write it as $-\frac{1}{u^2}u' = x$. Integrate both sides (with respect to x) to get $\frac{1}{u} = \frac{1}{2}x^2 + a$. Put in $x = 0$, $u = y' = \frac{1}{2}$ to get $2 = a$, so we have $\frac{1}{u} = \frac{1}{2}x^2 + 2 = \frac{x^2+4}{2}$, that is $y' = u = \frac{2}{x^2+4}$. Integrate to get $y = \int \frac{2\,dx}{x^2+4} = \tan^{-1}\left(\frac{x}{2}\right) + b$. Put in $x = 0$, $y = 2$ to get $2 = b$, and so the solution to the IVP is $y = \tan^{-1}\left(\frac{x}{2}\right) + 2$.

2: Use the substitution $y' = u$, $y'' = u\,u'$, where $u = u(y)$, to solve the following IVPs.

(a) $y\,y'' + (y')^2 = 0$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: Make the substitution $y' = u$, $y'' = u\,u'$. The DE becomes $yu\,u' + u^2 = 0$. This is linear since we can write it as $u' + \frac{1}{y}u = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$ and the solution is $u = \frac{1}{y}\int 0\,dy = \frac{a}{y}$. Put in $x = 1$, $y = 2$, $u = y' = 3$ to get $3 = \frac{a}{2}$ so $a = 6$ and the solution is $u = \frac{6}{y}$, that is $y' = \frac{6}{y}$. This DE is separable since we can write it as $y\,y' = 6$. Integrate both sides (with respect to x) to get $\frac{1}{2}y^2 = 6x + c$. Put in $x = 1$, $y = 2$ to get $2 = 6 + x$ so $c = -4$ and the solution is $\frac{1}{2}y^2 = 6x - 4$, that is $y = \pm\sqrt{12x - 8}$. Since $y(1) = 2$, we must use the $+$ sign, so $y = \sqrt{12x - 8}$.

(b) $y'' + (y')^2 = 2e^{-y}$ with $y(0) = 0$ and $y'(0) = 2$.

Solution: Make the substitution $y' = u$ so $y'' = u\,u'$. The DE becomes $u\,u' + u^2 = 2e^{-y}$. This is a Bernoulli equation since we can write it as $u' + u = 2e^{-y}\,u^{-1}$. Let $v = u^2$ so $v' = 2u\,u'$. Multiply the Bernoulli equation by $2u$ to get $2u\,u' + 2u^2 = 4e^{-y}$, and write this as $v' + 2v = 4e^{-y}$. This is linear. An integrating factor is $\lambda = e^{\int 2\,dy} = e^{2y}$ and the solution is $v = e^{-2y}\int 4e^{y} = e^{-2y}(4e^{y} + b)$. Put in $x = 0$, $y = 0$, $u = y' = 2$ and $v = u^2 = 4$ to get $4 = 4 + b$, so $b = 0$ and we have $v = 4e^{-y}$. Since $v = u^2 = (y')^2$, we have $(y')^2 = 4e^{-y}$ so $y' = \pm 2e^{-y/2}$. Since $y'(0) = 2$ we must use the $+$ sign, so $y' = 2e^{-y/2}$. This DE is separable since we can write it as $e^{y/2}y' = 2$. Integrate both sides to get $2e^{y/2} = 2x + c$. Put in $x = 0$ and $y = 0$ to get $2 = c$, so the solution is given by $2e^{y/2} = 2x + 2$. Solve for $y = y(x)$ to get $y = 2\ln(x + 1)$.

3: Use the method of reduction of order to solve each of the following.

(a) Solve the DE $x^3y'' + xy' - y = 0$, given that $y = x$ is one solution.

Solution: Let $y = xu$, $y' = u + xu'$, $y'' = 2u' + xu''$. Put this into the DE to get

$$0 = x^3y'' + xy' - y = 2x^3u' + x^4u'' + xu + x^2u' - xu = x^4u'' + (2x^3 + x^2)u' ,$$

and so, by dividing by x^4, we get $u'' + \left(\frac{2}{x} + \frac{1}{x^2}\right) u' = 0$. Let $u' = v$ and $u'' = v'$. Then the DE becomes $v' + \left(\frac{2}{x} + \frac{1}{x^2}\right) v = 0$. This is linear. An integrating factor is $\lambda = e^{\int \frac{2}{x}+\frac{1}{x^2}\,dx} = e^{2\ln x - \frac{1}{x}} = x^2e^{-1/x}$ and the solution is $v = \dfrac{e^{1/x}}{x^2}\displaystyle\int 0\,dx = \dfrac{ae^{1/x}}{x^2}$, that is $u' = \dfrac{ae^{1/x}}{x^2}$. Integrate to get $u = \displaystyle\int \frac{ae^{1/x}}{x^2}\,dx = -ae^{1/x} + b$. We only need one more independent solution, so take $a = -1$ and $b = 0$ to get $u = e^{1/x}$. Thus $y = xu = xe^{1/x}$ is a second independent solution. The general solution is $y = Ax + Bxe^{1/x}$.

(b) Solve the IVP $x^2y'' + 3xy' + y = 0$ with $y(1) = 2$ and $y'(1) = 3$ given that $y = \frac{1}{x}$ is a solution to the DE.

Solution: Let $y = \frac{1}{x}\,u$ so $y' = -\frac{1}{x^2}\,u + \frac{1}{x}\,u'$ and $y'' = \frac{2}{x^3}\,u - \frac{2}{x^2}\,u' + \frac{1}{x}\,u''$. Put this into the DE to get

$$0 = x^2y'' + 3xy' + y = \tfrac{2}{x}u - 2u' + xu'' - \tfrac{3}{x}u + 3u' + \tfrac{1}{x}u = xu'' + u' .$$

Let $u' = v$ and $u'' = v'$, and we get $xv' + v = 0$. This is linear since we can write it as $v' + \frac{1}{x}v = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and the solution is $v = \frac{1}{x}\int 0 = \frac{a}{x}$, that is $u' = \frac{a}{x}$. Integrate to get $u = a\ln x + b$. We only need one more independent solution so we take $a = 1$, $b = 0$ to get $u = \ln x$. Thus $y = \frac{1}{x}\,u = \frac{\ln x}{x}$ is a second independent solution. The general solution is $y = \frac{A + B\ln x}{x}$. To get $y(1) = 2$ we need $2 = a$, so we have $y = \frac{2 + B\ln x}{x}$. Note that $y' = \frac{B - 2 - B\ln x}{x^2}$, so to get $y'(1) = 3$ we need $3 = B - 2$, so $B = 5$ and the solution to the IVP is $y = \frac{2 + 5\ln x}{x}$.

4: Use the method of variation of parameters to solve each of the following.

(a) Solve the DE $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$ given that $y = x$ and $y = xe^x$ are solutions to the associated homogeneous DE.

Solution: We can write the DE as $y'' - \frac{x+2}{x}\,y' + \frac{x+2}{x^2}\,y = 2x$. Let $y_p = xu + xe^xv$ where $xu' + xe^xv' = 0$, that is $u' + e^xv' = 0$ (1). Putting this into the DE and simplifying gives $u' + (x+1)e^xv' = 2x$ (2). Multiply equation (1) by $(x+1)$ and subtract (2) to get $xu' = -2x$, that is $u' = -2$. Integrate to get $u = \int -2\,dx = -2x$ (we can take the arbitrary constant to be 0). Subtract (1) from (2) to get $xe^xv' = 2x$, that is $v' = 2e^{-x}$. Integrate to get $v = -2e^{-x}$. Thus a particular solution to the given (non-homogeneous) DE is $y_p = xu + xe^xv = -2x^2 - 2x = -2x(x+1)$. The general solution is $y = ax + bxe^x - 2x(x+1)$.

(b) Solve the DE $xy'' - (1+x)y' + y = x^2e^{2x}$ given that $y = 1 + x$ and $y = e^x$ are solutions to the associated homogeneous DE.

Solution: We can write the DE as $y'' - \frac{1+x}{x}\,y' + \frac{1}{x}\,y = xe^{2x}$. Let $y_p = (1+x)u + e^xv$ where u and v satisfy $(1+x)u' + e^xv' = 0$ (1). Putting this into the DE and simplifying gives $u' + e^xv' = xe^{2x}$ (2). Subtract (2) from (1) to get $xu' = -xe^{2x}$ so $u' = -e^{2x}$. Integrate to get $u = \int -e^{2x}\,dx = -\frac{1}{2}e^{2x}$ (we can take the arbitrary constant to be 0). Multiply (2) by $(1+x)$ and subtract (1) to get $xe^xv' = x(1+x)e^{2x}$, that is $v' = (1+x)e^x$. Integrate to get $v = \int (1+x)e^x\,dx = xe^x$. Thus a particular solution to the given (non-homogeneous) DE is $y_p = (1+x)u + e^xv = -\frac{1}{2}(1+x)e^{2x} + xe^{2x} = \frac{1}{2}(x-1)e^{2x}$. The general solution is $y = a(1+x) + be^x + \frac{1}{2}(x-1)e^{2x}$.

5: Consider the IVP $y'' = y\,y'$ with $y(0) = 1$ and $y'(0) = 1$.

(a) Find the exact solution $y = f(x)$ to the given IVP.

Solution: Make the substitution $y' = u$, $y'' = u\,u'$, where $u = u(y)$. The DE becomes $u\,u' = yu$, that is $u' = y$. Integrate both sides (with respect to y) to get $u = \frac{1}{2}y^2 + a$. Put in $x = 0$, $y = 1$, $u = y' = 1$ to get $1 = \frac{1}{2} + a$ so $a = \frac{1}{2}$ and the solution is $u = \frac{1}{2}y^2 + \frac{1}{2} = \frac{y^2+1}{2}$, that is $y' = \frac{y^2+1}{2}$. This is separable as we can write it as $\frac{y'}{y^2+1} = \frac{1}{2}$. Integrate (with respect to x) to get $\tan^{-1} y = \frac{1}{2}x + b$. Put in $x = 0$, $y = 1$ to get $\frac{\pi}{4} = b$ and so the solution is $\tan^{-1} y = \frac{1}{2}x + \frac{\pi}{4}$, that is $y = f(x) = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$.

(b) Use Euler's method with step size $\Delta x = 0.2$ to approximate $f(1)$.

Solution: The DE can be written as $y'' = F(x, y, y')$ where $F(x, y, z) = yz$. We let $\Delta x = 0.2$ and start with $x_0 = 0$, $y_0 = 1$ and $z_0 = 1$, and then for $k \geq 0$ we let $x_{k+1} = x_k + \Delta x$, $y_{k+1} = y_k + z_k \Delta x$ and $z_{k+1} = z_k + F(x_k, y_x, z_k)\Delta x = z_k + y_k z_k \Delta x$. We make a table showing the values of x_k, y_k, z_k and $F(x_k, y_k, z_k) = y_k z_k$.

k	x_k	y_k	z_k	$y_k z_k$
0	0.0	1	1	1
1	0.2	1.2	1.2	1.44
2	0.4	1.44	1.488	2.14272
3	0.6	1.7376	1.916544	3.3301869
4	0.8	2.1209088	2.5825814	5.4774196
5	1.0	2.6374251		

Thus $f(1) \cong y^5 \cong 2.6$ (this is not a very good approximation, as you can check using part (a)).