MATH 218 Differential Equations, Solutions to Assignment 3

1: (a) The amount $A(t)$ of a radioactive substance satisfies the DE $A'(t) = k\,A(t)$ for some constant $k < 0$. The substance has a half-life of 10 seconds, which means that $A(10) = \frac{1}{2}\,A(0)$. If $A(5) = 100$ then find the exact time t at which $A(t) = 20$.

Solution: This DE is linear, since we can write it in the form $A' - kA = 0$. An integrating factor is $\lambda = e^{\int -k\,dt} = e^{-k\,t}$ and the general solution is $A(t) = e^{k\,t}\int 0\,dt = c\,e^{kt}$. Note that $A(0) = c$, so c is the initial amount. Since the half-life is 10, we have

$$A(10) = \tfrac{1}{2}\,c \Longrightarrow c\,e^{10k} = \tfrac{1}{2}\,c \Longrightarrow e^{10k} = \tfrac{1}{2} \Longrightarrow 10k = \ln\tfrac{1}{2} = -\ln 2 \Longrightarrow k = -\tfrac{1}{10}\ln 2\,.$$

and so $A(t) = c\,e^{-(t/10)\ln 2} = c\,2^{-t/10}$. Also, we have $A(5) = 100 \Longrightarrow c\,2^{-1/2} = 100 \Longrightarrow c = 100\sqrt{2}$, and so $A(t) = \big(100\sqrt{2}\big)\,2^{-t/10}$. Finally, we have

$$\begin{aligned} A(t) = 20 \iff & \big(100\sqrt{2}\big)\,2^{-t/10} = 20 \\ \iff & 2^{t/10} = \frac{100\sqrt{2}}{20} = 5\sqrt{2} = \sqrt{50} \\ \iff & \frac{t}{10} = \log_2\sqrt{50} = \tfrac{1}{2}\log_2 50 \\ \iff & t = 5\log_2 50\,. \end{aligned}$$

(b) A murder victim is found in a room of constant teperature 25° C. At the time of murder, we assume that the victim's body temperature was 37° C. At 2:00 pm, the body temperature is measured to be 31° C and at 5:00 pm, it is measured to be 29° C. Determine the time of death, assuming that the temperature $T = T(t)$ of the body at time t satisfies Newton's Law of Cooling, so that $T' = -k(T-25)$ for some constant $k > 0$.

Solution: Let t hours be the time elapsed since 2:00 pm (so $t = 0$ at 2:00). The DE $T' = -k(T-25)$ is linear since we can write it as $T' + kT = 25k$. An integrating factor is $I = e^{\int k\,dt} = e^{kt}$ and the solution is

$$T = e^{-kt}\int 25k\,e^{kt}\,dt = e^{-kt}\big(25\,e^{kt} + c\big) = 25 + c\,e^{-kt}\,.$$

To get $T(0) = 31$ we need $25 + c = 31$ so $c = 6$, and so we have

$$T(t) = 25 + 6\,e^{-kt}\,.$$

Also we have $T(3) = 29 \Longrightarrow 25 + 6\,e^{-3k} = 29 \Longrightarrow 6\,e^{-3k} = 4 \Longrightarrow -3k = \frac{2}{3} \Longrightarrow k = -\frac{1}{3}\ln\frac{2}{3} = \frac{1}{3}\ln\frac{3}{2}$. Finally, $T(t) = 37 \iff 25 + 6\,e^{-kt} = 37 \iff 6\,e^{-kt} = 12 \iff e^{-kt} = 2 \iff -kt = \ln 2$ so $t = -\frac{\ln 2}{k} = -\frac{3\ln 2}{\ln\frac{3}{2}} \cong -5.13$. Thus the victim died about 5 hours before 2:00, that is at about 9:00 am.

2: A tank initially contains 12 L of brine (salt-water) with a salt-concentration of 6 g/L. Brine with a salt-concentration of 1 g/L enters the tank at a rate of 2 L/min. The tank is kept well-mixed, and brine drains from the tank at a rate of $\frac{1}{2}$ L/min. Determine the time at which the salt-concentration in the tank reaches 3 g/L.

Solution: Let $V(t)$ L be the volume of brine in the tank at time t min, and let $S(t)$ g be the amount of salt in the tank at time t min. Also, write $V_0 = 12$, $c_0 = 6$, $c_{\text{in}} = 1$, $r_{\text{in}} = 2$ and $r_{\text{out}} = \frac{1}{2}$. Since the initial volume is $V_0 = 12$, and the volume is changing at the rate $V' = r_{\text{in}} - r_{\text{out}} = 2 - \frac{1}{2} = \frac{3}{2}$, we have

$$V(t) = \tfrac{3}{2}\,t + 12\,,$$

and so $S(t)$ satisfies the IVP $S' = r_{\text{in}}c_{\text{in}} - r_{\text{out}}\,\frac{S}{V} = 1\cdot 2 - \frac{1}{2}\cdot\frac{S}{\frac{3}{2}\,t+12} = 2 - \frac{S}{3(t+8)}$ with $S(0) = c_0V_0 = 6\cdot 12 = 72$. This DE is linear since we can write it as

$$S' + \frac{1}{3(t+8)}\,S = 2\,.$$

An integrating factor is $I = e^{\int \frac{1}{3}(t+8)\;dt} = e^{\frac{1}{3}\ln(t+8)} = (t+8)^{1/3}$, and the solution is

$$S = (t+8)^{-1/3}\int 2(t+8)^{1/3}\,dt = (t+8)^{-1/3}\Big(\tfrac{3}{2}(t+8)^{4/3} + c\Big) = \tfrac{3}{2}(t+8) + c\,(t+8)^{-1/3}\,.$$

We have $S(0) = 72 \Longrightarrow 12 + \frac{1}{2}\,c = 72 \Longrightarrow c = 120$, and so

$$S(t) = \tfrac{3}{2}(t+8) + 120(t+8)^{-1/3}\,.$$

The concentration $C(t)$ g/L of salt in the tank is given by

$$C(t) = \frac{S(t)}{V(t)} = \frac{\frac{3}{2}(t+8) + 120(t+8)^{-1/3}}{\frac{3}{2}(t+8)} = 1 + 80(t+8)^{-4/3}\,,$$

andwe have $C(t) = 3 \iff 1 + 80(t+8)^{-4/3} = 3 \iff 80(t+8)^{-4/3} = 2 \iff (t+8)^{-4/3} = \frac{1}{40} \iff (t+8) = 40^{3/4} \iff t = 40^{3/4} - 8 \cong 7.9$. Thus the concentration reaches 3 g/L after about 7.9 minutes.

3: A tank, in the shape of a lower-hemisphere of radius 1 m, is initially filled with water. Water drains through a circular hole of diameter 5 cm at the bottom of the tank. When the depth of water in the tank is equal to y m, the water flows through the hole at a speed of $4\sqrt{y}$ m/s. Determine the time it takes for the depth of the water in the tank to reach 25 cm.

Solution: The front view of the tank is shaped like the bottom half of the circle $x^2 + (y-1)^2 = 1$, and the right half of this circle is given by $x = \sqrt{1-(y-1)^2} = \sqrt{2y - y^2}$. The horizontal cross-section of the tank at height y is a circle of radius $r = x = \sqrt{2y-y^2}$, and so a slice of thickness Δy has volume $\Delta V \cong \pi(2y - y^2)\Delta y$. Thus for a small time interval Δt we have $\frac{\Delta V}{\Delta t} \cong \pi(2y-y^2)\frac{\Delta y}{\Delta t}$. As $\Delta t \to 0$ we get

$$V' = \pi(2y - y^2)\, y' \,.$$

On the other hand, since the water flows through a hole of area $a = \pi(.025)^2 = \pi\left(\frac{1}{40}\right)^2 = \frac{\pi}{1600}$ m^2 at a speed $v = 4\sqrt{y}$ m/s, we also have

$$V' = -av = -\tfrac{\pi}{400}\sqrt{y}\,.$$

Equating these two expressions for V' we obtain

$$\pi(2y-y^2)\, y' = -\tfrac{\pi}{400}\sqrt{y}\,.$$

This DE is separable as we can write it as $(2y^{1/2} - y^{3/2})y' = \frac{-1}{400}$. Integrate both sides, using the substitution $y = y(t)$ on the left, to get $\displaystyle\int 2y^{1/2} - y^{3/2}\, dy = \int -\tfrac{1}{400}\, dt$ which gives

$$\tfrac{4}{3}y^{3/2} - \tfrac{2}{5}\, y^{5/2} = -\tfrac{1}{400}\, t + c\,.$$

To get $y(0) = 1$ we need $\frac{4}{3} - \frac{2}{5} = c$ so $c = \frac{14}{15}$ and so we have $\frac{4}{3}\, y^{3/2} - \frac{2}{5}\, y^{5/2} = -\frac{1}{400}\, t + \frac{14}{15}$. Thus we have $y(t) = \frac{1}{4} \iff \frac{1}{400}\, t = \frac{14}{15} - \frac{4}{3}\cdot\frac{1}{8} + \frac{2}{5}\cdot\frac{1}{32} = \frac{14}{15} - \frac{1}{6} + \frac{1}{80} \iff t = \frac{1120}{3} - \frac{200}{3} + 5 = \frac{935}{3}$.

4: In a chemical reaction, 2 g of substance A reacts with 1 g of substance B to produce 3 g of substance C. Suppose that 4 g of substance A and 3 g of substance B are combined at time $t = 0$ min. Let $a(t)$, $b(t)$ and $c(t)$ be the amounts, in grams, of the three substances, and suppose that

$$c'(t) = 3\,a(t)\,b(t)\,.$$

Find a formula for $c(t)$, and find the time at which 3 g of substance C has been produced.

Solution: To produce c g of substance C we must use up $\frac{2}{3}\,c$ g of substance A and $\frac{1}{3}\,c$ g of substance B, and so $a(t) = 4 - \frac{2}{3}\,c(t)$ and $b(t) = 3 - \frac{1}{3}\,c(t)$. Thus $c' = 3ab = 3\left(4 - \frac{2}{3}\,c\right)\left(3 - \frac{1}{3}\,c\right) = (12 - 2c)\left(3 - \frac{1}{3}\,c\right)$ so

$$3\,c' = 2(6-c)(9-c)\,.$$

This DE is separable since we can write it as $\dfrac{3}{(6-c)(9-c)}\,c' = 2$. Integrate both sides to get

$$\int \frac{3\,dc}{(6-c)(9-c)} = \int 2\,dt = 2t + d\,.$$

We have $\displaystyle\int \frac{3\,dc}{(6-c)(9-c)} = \int \frac{1}{6-c} - \frac{1}{9-c}\,dc = -\ln(6-c) + \ln(9-c) + \text{const} = \ln\frac{9-c}{6-c} + \text{const}$, and so we have $\ln\dfrac{9-c}{6-c} = 2t + d$. To get $c(0) = 0$ we need $\ln\frac{3}{2} = d$ and so

$$\ln\frac{9-c}{6-c} = 2t + \ln\tfrac{3}{2} \quad (1)$$

Thus $\dfrac{9-c}{6-c} = e^{2t + \ln\frac{3}{2}} = \frac{3}{2}\,e^{2t} \Longrightarrow 18 - 2c = 18e^{2t} - 3ce^{2t} \Longrightarrow c(3e^{2t} - 2) = 18e^{2t} - 18 \Longrightarrow c = \dfrac{18(e^{2t} - 1)}{3e^{2t} - 2}$

Finally, put $c = 3$ into equation (1) to get $\ln 2 = 2t + \ln\frac{3}{2} \Longrightarrow 2t = \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3} \Longrightarrow t = \frac{1}{2}\ln\frac{4}{3}\,.$

5: A tank, in the shape of an inverted pyramid (square at the top with a sharp point at the bottom) with top square side length a meters and height b meters, is initially empty. Water flows in at a rate of r m^3/s. Water evaporates from the surface of the water in the tank at a rate of kA m^3/s where A m^2 is the area of the surface. Let $y = y(t)$ be the depth (in meters) of the water in the tank at time t (in seconds).

(a) Show that $y = y(t)$ satisfies the IVP $a^2y^2\,y' = rb^2 - ka^2y^2$ with $y(0) = 0$.

Solution: The horizontal cross-section of the water in the tank at position y is a square with sides of length x with $\frac{x}{y} = \frac{a}{b}$ (by similar triangles) so $x = \frac{a}{b}\,y$, and so the area of the cross-section at y is $A(y) = x^2 = \frac{a^2y^2}{b^2}$. A horizontal slice of water at position y and thickness Δy has volume $\Delta V = A(y)\Delta y = \frac{a^2y^2}{b^2}\Delta y$, and so we have $\frac{\Delta V}{\Delta t} = \frac{a^2y^2}{b^2}\,\frac{\Delta y}{\Delta t}$. Letting $\Delta t \to 0$, we obtain

$$V' = \frac{a^2y^2}{b^2}\,y'\,.$$

On the other hand, water enters the tank at the rate r and leaves the tank at the rate $kA(y) = k\frac{a^2y^2}{b^2}$ and so we have

$$V' = r - \frac{ka^2y^2}{b^2}\,.$$

Equating the above two expressions for V' gives us the DE $\frac{a^2y^2}{b^2}\,y' = r - \frac{ka^2y^2}{b^2}$. Multiplying by b^2 gives

$$a^2y^2\,y' = rb^2 - ka^2y^2\,.$$

(b) Find a formula for the equilibrium (steady-state) depth, and find conditions on a, b, r and k in order that the tank never overflows.

Solution: The equilibrium depth occurs when $y' = 0$. Putting $y' = 0$ into the DE from part (a) gives $rb^2 = ka^2y^2$ so $y^2 = \frac{rb^2}{ka^2}$. Thus the equilibrium depth is

$$y = \frac{\sqrt{r}\,b}{\sqrt{k}\,a}\,.$$

The tank will overflow eventually when $\frac{\sqrt{r}\,b}{\sqrt{k}\,a} > b$, that is when $\sqrt{r} > \sqrt{k}\,a$, or equivalently when $r > ka^2$.

(c) Find a formula for the time t when the depth y reaches half of the equilibrium depth.

Solution: We shall need a formula for $\displaystyle\int \frac{y^2\,dy}{A^2 - B^2y^2}$. We find this by partial fractions. We have

$$\int \frac{y^2\,dy}{A^2 - B^2y^2} = \int -\frac{1}{B^2} + \frac{\frac{A^2}{B^2}}{A^2 - B^2y^2}\,dy = \int -\frac{1}{B^2} + \frac{\frac{A}{2B^2}}{A + By} + \frac{\frac{A}{2B^2}}{A - By}\,dy = -\frac{1}{B}\,y + \frac{A}{2B^3}\ln\left|\frac{A + By}{A - By}\right| + c\,.$$

Now we solve the DE from part (a). It is linear since we can write it as $\frac{a^2y^2}{rb^2 - ka^2}\,y' = 1$. Integrate both sides, using the above formula with $A = \sqrt{r}\,b$ and $B = \sqrt{k}\,a$, to get

$$\int \frac{a^2y^2\,dy}{rb^2 - ka^2y^2} = \int 1\,dt$$

$$-\frac{1}{k}\,y + \frac{\sqrt{r}\,b}{2k\sqrt{k}\,a}\ln\left|\frac{\sqrt{r}\,b + \sqrt{k}\,ay}{\sqrt{r}\,b - \sqrt{k}\,ay}\right| = t + c\,.$$

To get $y(0) = 0$ we need $0 = c$, so we have $t = -\frac{1}{k}\,y + \frac{\sqrt{r}\,b}{2k\sqrt{k}\,a}\ln\left|\frac{\sqrt{r}\,b + \sqrt{k}\,ay}{\sqrt{r}\,b - \sqrt{k}\,ay}\right|$. The water reaches half the equilibrium depth when $y = \frac{\sqrt{r}\,b}{2\sqrt{k}\,a}$, that is when

$$t = -\frac{\sqrt{r}\,b}{2k\sqrt{k}\,a} + \frac{\sqrt{r}\,b}{2k\sqrt{k}\,a}\ln\left(\frac{\sqrt{r}\,b + \frac{\sqrt{r}\,b}{2}}{\sqrt{r}\,b - \frac{\sqrt{r}\,b}{2}}\right) = \frac{\sqrt{r}\,b}{2k\sqrt{k}\,a}\big(\ln 3 - 1\big)\,.$$