MATH 218 Differential Equations, Assignment 3 Due Tues Sept 30

1: (a) The amount $A(t)$ of a radioactive substance satisfies the DE $A'(t) = k\,A(t)$ for some constant $k < 0$. The substance has a half-life of 10 seconds, which means that $A(10) = \frac{1}{2}\,A(0)$. If $A(5) = 100$ then find the exact time t at which $A(t) = 20$.

(b) A murder victim is found in a room of constant teperature 25° C. At the time of murder, we assume that the victim's body temperature was 37° C. At 2:00 pm, the body temperature is measured to be 31° C and at 5:00 pm, it is measured to be 29° C. Determine the time of death, assuming that the temperature $T = T(t)$ of the body at time t satisfies Newton's Law of Cooling, so that $T' = -k(T - 25)$ for some constant $k > 0$.

2: A tank initially contains 12 L of brine (salt-water) with a salt-concentration of 6 g/L. Brine with a salt-concentration of 1 g/L enters the tank at a rate of 2 L/min. The tank is kept well-mixed, and brine drains from the tank at a rate of $\frac{1}{2}$ L/min. Determine the time at which the salt-concentration in the tank reaches 3 g/L.

3: A tank, in the shape of a lower-hemisphere of radius 1 m, is initially filled with water. Water drains through a circular hole of diameter 5 cm at the bottom of the tank. When the depth of water in the tank is equal to y m, the water flows through the hole at a speed of $4\sqrt{y}$ m/s. Determine the time it takes for the depth of the water in the tank to reach 25 cm.

4: In a chemical reaction, 2 g of substance A reacts with 1 g of substance B to produce 3 g of substance C. Suppose that 4 g of substance A and 3 g of substance B are combined at time $t = 0$ min. Let $a(t)$, $b(t)$ and $c(t)$ be the amounts, in grams, of the three substances, and suppose that

$$c'(t) = 3\,a(t)\,b(t)\,.$$

Find a formula for $c(t)$, and find the time at which 3 g of substance C has been produced.

5: A tank, in the shape of an inverted pyramid (square at the top with a sharp point at the bottom) with top square side length a meters and height b meters, is initially empty. Water flows in at a rate of r m^3/s. Water evaporates from the surface of the water in the tank at a rate of kA m^3/s where A m^2 is the area of the surface. Let $y = y(t)$ be the depth (in meters) of the water in the tank at time t (in seconds).

(a) Show that $y = y(t)$ satisfies the IVP $a^2y^2\,y' = rb^2 - ka^2y^2$ with $y(0) = 0$.

(b) Find a formula for the equilibrium (steady-state) depth, and find conditions on a, b, r and k in order that the tank never overflows.

(c) Find a formula for the time t when the depth y reaches half of the equilibrium depth.