MATH 218 Differential Equations, Solutionss to Assignment 1

1: Find each of the following integrals.

(a) $\displaystyle\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\,dx$

Solution: Make the substitution $u=\cos x$ so $du=-\sin x\;dx$. Then

$$\begin{aligned}\int_0^{\pi/3}\frac{\sin^3 x}{\cos^2 x}\,dx &= \int_0^{\pi/3}\frac{(1-\cos^2 x)}{\cos^2 x}\,\sin x\;dx=\int_1^{1/2}-\frac{1-u^2}{u^2}\,du=\int_1^{1/2}-\tfrac{1}{u^2}+1\;du\\ &=\Big[\tfrac{1}{u}+u\Big]_1^{1/2}=\big(2+\tfrac12\big)-(1+1)=\tfrac12\,.\end{aligned}$$

(b) $\displaystyle\int_0^3\frac{x^2\,dx}{(x+1)^{3/2}}$

Solution: Make the substitution $u=x+1$ so $x=u-1$ and $dx=du$. Then

$$\begin{aligned}\int_0^3\frac{x^2\,dx}{(x+1)^{3/2}} &= \int_1^4\frac{(u-1)^2}{u^{3/2}}\,du=\int_1^4\frac{u^2-2u+1}{u^{3/2}}\,du=\int_1^4 u^{1/2}-2u^{-1/2}+u^{-3/2}\;du\\ &=\Big[\tfrac23\,u^{3/2}-4u^{1/2}-2u^{-1/2}\Big]_1^4=\big(\tfrac{16}{3}-8-1\big)-\big(\tfrac23-4-2\big)=\tfrac53\,.\end{aligned}$$

(c) $\displaystyle\int_0^\pi(3x+1)\cos(x/2)\,dx$

Solution: Integrate by parts using $u=3x+1$, $du=3\,dx$, $v=2\sin(x/2)$ and $dv=cos(x/2)\,dx$ to get

$$\begin{aligned}\int_0^\pi(3x+1)\cos(x/2)\,dx &= \Big[2(3x+1)\sin(x/2)-\int 6\sin(x/2)\,dx\Big]_0^\pi\\ &=\Big[2(3x+1)\sin(x/2)+12\cos(x/2)\Big]_0^\pi\\ &=2(3\pi+1)-12=6\pi-10\,.\end{aligned}$$

(d) $\displaystyle\int_0^{\pi/2} e^{2x}\cos x\;dx$

Solution: Let $I=\displaystyle\int e^{2x}\cos x\;dx$. Integrate by parts twice, first using $u_1=e^{2x}$, $du_1=2e^{2x}\,dx$, $v_1=\sin x$ and $dv_1=\cos x\,dx$, and then using $u_2=2e^{2x}$, $du_2=4e^{2x}\,dx$, $v_2=-\cos x$ and $dv_2=\sin x\,dx$ to get

$$\begin{aligned}\int e^{2x}\cos x\;dx &= u_1v_1-\int v_1du_1=e^{2x}\sin x-\int 2e^{2x}\sin x\,dx=e^{2x}\sin x-\Big(u_2v_2-\int v_2\,du_2\Big)\\ &=e^{2x}\sin x-\Big(-2e^{2x}\cos x+\int 4e^{2x}\cos x\,dx\Big)=e^{2x}\big(\sin x+2\cos x\big)-4I\,.\end{aligned}$$

and so $5I=e^{2x}\big(\sin x+2\cos x\big)+c$, that is $I=\frac15\,e^{2x}\big(\sin x+2\cos x\big)+d$. Thus

$$\int_0^{\pi/2}e^{2x}\cos x\;dx=\Big[\tfrac15\,e^{2x}\big(\sin x+2\cos x\big)\Big]_0^{\pi/2}=\tfrac15\,e^\pi-\tfrac25$$

2: Find each of the following integrals.

(a) $\displaystyle\int_0^{\sqrt{3}} \frac{x^2\,dx}{\sqrt{4-x^2}}$

Solution: Make the substitution $2\sin\theta = x$, $2\cos\theta = \sqrt{4-x^2}$, $2\cos\theta\,d\theta = dx$ to get

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{x^2\,dx}{\sqrt{4-x^2}} &= \int_0^{\pi/3} \frac{4\sin^2\theta\cdot 2\cos\theta\,d\theta}{2\cos\theta} = \int_0^{\pi/3} 4\sin^2\theta\,d\theta = \int_0^{\pi/3} 2-2\cos 2\theta\ d\theta \\ &= \Big[2\theta - \sin 2\theta\Big]_0^{\pi/3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\,.\end{aligned}$$

(b) $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+2x+4)^{3/2}}$

Solution: Make the substitution $\sqrt{3}\tan\theta = x+1$, $\sqrt{3}\sec\theta = \sqrt{x^2+2x+4}$, $\sqrt{3}\sec^2\theta\,d\theta = dx$ to get

$$\int_0^{\infty} \frac{dx}{(x^2+2x+4)^{3/2}} = \int_{\pi/6}^{\pi/2} \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\sqrt{3}\sec^3\theta} = \int_{\pi/6}^{\pi/2} \tfrac{1}{3}\cos\theta\,d\theta = \Big[\tfrac{1}{3}\sin\theta\Big]_{\pi/6}^{\pi/2} = \tfrac{1}{3}\left(1-\tfrac{1}{2}\right) = \tfrac{1}{6}\,.$$

(c) $\displaystyle\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx$

Solution: We use long division to obtain

$$\begin{array}{rr} & x-1 \\ \cline{2-2} x^3+4x^2+3x & \big)\ x^4+3x^3+0\,x^2+0x+6 \\ & \underline{x^4+4x^3+3x^2}\qquad\qquad \\ & -x^3-3x^2+0x+6 \\ & \underline{-x^3-4x^2-3x}\quad \\ & x^2+3x+6 \end{array}$$

This shows that $\dfrac{x^4+3x^3+6}{x^3+4x^2+3x} = x-1+\dfrac{x+3x+6}{x^3+4x^2+3x}$. Note that $x^3+4x^2+3x = x(x+1)(x+3)$. In order to get $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+3} = \dfrac{x+3x+6}{x(x+1)(x+3)}$ for all x we need $A(x+1)(x+3)+Bx(x+3)+Cx(x+1) = x^2+3x+6$ for all x. Put in $x=0$ to get $3A=6$ so $A=2$, put in $x=-1$ to get $-2B=4$ so $B=-2$, and put in $x=-3$ to get $6C=6$ so $C=1$. Thus

$$\begin{aligned}\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx &= \int_1^3 x-1+\frac{2}{x}-\frac{2}{x+1}+\frac{1}{x+3}\ dx \\ &= \Big[\tfrac{1}{2}x^2 - x + 2\ln x - 2\ln(x+1) + \ln(x+3)\Big]_1^3 \\ &= \left(\tfrac{9}{2} - 3 + 2\ln 3 - 2\ln 4 + \ln 6\right) - \left(\tfrac{1}{2} - 1 - 2\ln 2 + \ln 4\right) \\ &= 2 + 3\ln 3 - 3\ln 2 = 2 + 3\ln\tfrac{3}{2}\,.\end{aligned}$$

(d) $\displaystyle\int_1^2 \frac{x^3+2}{x^5+2x^3+x}\,dx$

Solution: To begin with, we find $\displaystyle\int \frac{dx}{(x^2+1)^2}$, which we need later. Let $\tan\theta = x$ so that $\sec\theta = \sqrt{x^2+1}$ and $\sec^2\theta\,d\theta = dx$. Then

$$\begin{aligned}\int \frac{dx}{(x^2+1)^2} &= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\;d\theta \\ &= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + c = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\cos\theta + c = \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\,\frac{x}{x^2+1} + c\,.\end{aligned}$$

Note that $x^5+2x^3+x = x(x^2+1)^2$. In order to get $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} = \dfrac{x^3+2}{x(x^2+1)^2}$ we need $A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+F)(x^2+1) = x^3+2$. Equate coefficients to get the 5 equations $A+B=0$, $C=1$, $2A+B+D=0$, $C+E=0$ and $A=2$. Solve these to get $A=2$, $B=-2$, $C=1$, $D=-2$ and $E=-1$, and so

$$\begin{aligned}\int_1^2 \frac{x^3+2}{x^5+2x^3+x}\,dx &= \int_1^2 \frac{2}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x}{(x^2+1)^2} - \frac{1}{(x^2+1)^2}\;dx \\ &= \left[2\ln x - \ln(x^2+1) + \tan^{-1}x + \frac{1}{x^2+1} - \left(\tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\frac{x}{x^2+1}\right)\right]_1^2 \\ &= \left[2\ln x - \ln(x^2+1) + \tfrac{1}{2}\tan^{-1}x + \frac{2-x}{2(x^2+1)}\right]_1^2 \\ &= \left(2\ln 2 - \ln 5 + \tfrac{1}{2}\tan^{-1}2\right) - \left(-\ln 2 + \tfrac{\pi}{8} + \tfrac{1}{4}\right) \\ &= \ln\tfrac{8}{5} + \tfrac{1}{2}\tan^{-1}2 - \tfrac{\pi}{8} - \tfrac{1}{4}\,.\end{aligned}$$

3: (a) Verify that $y = x\sin x$ is a solution of the DE $y\,(y''+y) = x\sin 2x$.

Solution: We have $y' = \sin x + x\cos x$ and $y'' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$ and so

$$\begin{aligned}y\,(y''+y) &= (x\sin x)(2\cos x - x\sin x + x\sin x) \\ &= (x\sin x)(2\cos x) \\ &= x(2\sin x\cos x) \\ &= x\sin 2x\,.\end{aligned}$$

(b) Find all the solutions of the form $y = ax^2+bx+c$ to the DE $\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1$.

Solution: For $y = ax^2+bx+c$ we have $y' = 2ax+b$, so

$$\begin{aligned}\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1 &\iff \big(y'(x)\big)^2 + 4x - 3y(x) - x^2 - 1 = 0 \\ &\iff (2ax+b)^2 + 4x - 3(ax^2+bx+c) - x^2 - 1 = 0 \\ &\iff (4a^2-3a-1)x^2 + (4ab+4-3b)x + (b^2-3c-1) = 0 \\ &\iff 4a^2-3a-1 = 0\;,\;\; 4ab+4 = 3b\;,\;\text{ and } b^2 = 3c+1\end{aligned}$$

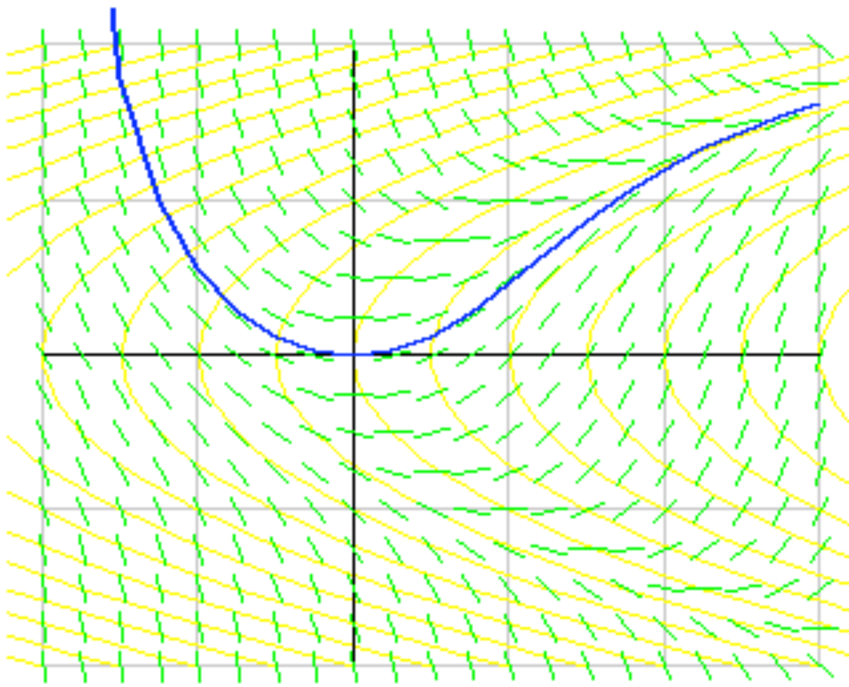
From $4a^2-3a-1=0$ we get $(4a+1)(a-1)=0$ and so $a = -\frac{1}{4}$ or $a=1$. When $= -\frac{1}{4}$, the equation $4ab+4=3b$ gives $-1+4=3b$ so $b=1$, and then the equation $b^2=3c+1$ gives $1=3c+1$ so $c=0$. When $a=1$, $4ab+4=3b$ gives $4b+4=3b$ so $b=-4$ and then $b^2=3c+1$ gives $16=3c+1$ so $c=5$. Thus there are two solutions, and they are $y = -\frac{1}{4}x^2+x$ and $y = x^2-4x+5$.

4: Consider the IVP $y' = x - y^2$ with $y(0) = 0$.

(a) Sketch the direction field for the given DE for $-2 \le x \le 3$ and $-2 \le y \le 2$.

(b) On the same grid, sketch the solution curve to the given IVP.

Solution: The isocline (curve of constant slope) $y' = m$ is the sideways parabola $m = x - y^2$, or $x = y^2 + m$. The isoclines are shown in yellow, the slope field is shown in green, and the solution curve with $y(0) = 0$ is shown in blue.



(c) Using a calculator, apply Euler's method with step size $\Delta x = 0.5$ to approximate the value of $f(3)$ where $y = f(x)$ is the solution to the given IVP.

Solution: We let $x_0 = 0$ and $y_0 = 0$, then for $k \ge 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x, y) = x - y^2$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	0	0	0
1	0.5	0	0.5
2	1.0	0.25	0.9375
3	1.5	0.71875	0.9833984375
4	2.0	1.210449219	0.534812688
5	2.5	1.477855563	0.315942935
6	3.0	1.635827030	

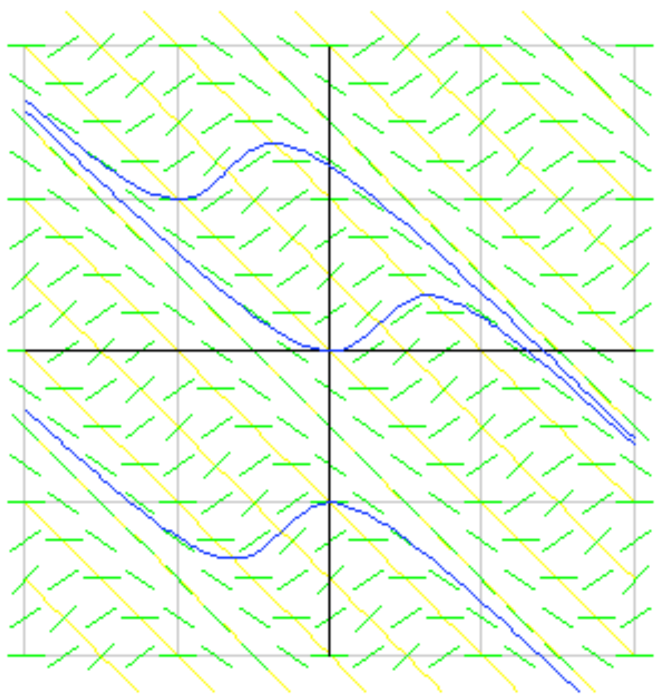
Thus we have $f(3) \cong y_6 \cong 1.8$.

5: Consider the IVP $y' = \sin\big(\pi(x+y)\big)$ with $y(-1) = 1$.

(a) Sketch the direction field for the given DE for $-2 \le x \le 2$ and $-2 \le y \le 2$.

(b) On the same grid, sketch the solution curves which pass through each of the points $(-1, 1)$, $(0, 0)$ and $(0, -1)$.

Solution: We have $y' = 0$ when $\sin\big(\pi(x+y)\big) = 0$, that is when $\pi(x+y) = k\pi$ for some integer k, or equivalently when $x+y = k$ for some integer k. Similarly, we have $y' = 1$ when $x+y = k+\frac{1}{2}$, and $y' = -1$ when $x+y = k - \frac{1}{2}$, and $y' = \frac{1}{2}$ when $x+y = k+\frac{1}{6}$ or $k+\frac{5}{6}$, and $y' = -\frac{1}{2}$ when $x+y = k-\frac{1}{6}$ or $k-\frac{5}{6}$. The isoclines $y' = 0, \pm\frac{1}{2}, \pm 1$ are shown in yellow, the direction field is shown in green, and the solution curves are shown in blue.



(c) Using a calculator, apply Euler's method with step size $\Delta x = 0.2$ to approximate the value of $f(0)$ where $y = f(x)$ is the solution to the given IVP.

Solution: We let $x_0 = -1$ and $y_0 = 1$, then for $k \ge 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x, y) = \sin\big(\pi(x + +y)\big)$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	-1	1	0
1	-0.8	1	0.5877852524
2	-0.6	1.1117557050	0.9984792328
3	-0.4	1.317252897	0.2570396643
4	-0.2	1.368660830	-0.5054156715
5	0	1.267577696	

Thus we have $f(0) \cong y_5 \cong 1.3$.