MATH 218 Differential Equations, Solutions to Assignment 11

**1:** (a) Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and draw the phase diagram.

Solution: Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Then $\left|A - rI\right| = \begin{vmatrix} 1-r & 2 \\ 1 & -r \end{vmatrix} = r^2 - r - 2 = (r+1)(r-2)\,.$

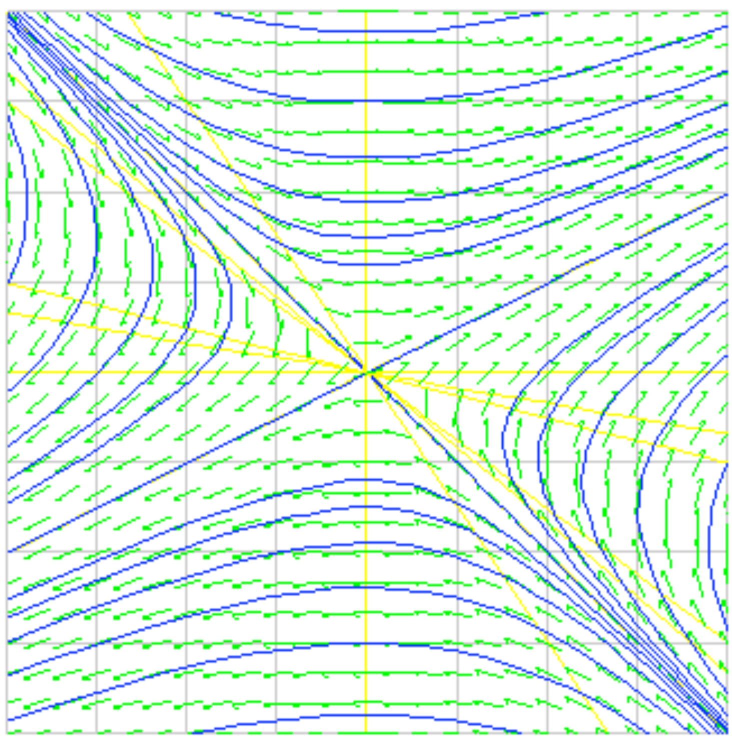
When $r = -1$ we have $A - rI = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

When $r = 2$ we have $A - rI = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + be^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \,.$$

To sketch the phase portrait, note that the isoclines are given by $m = \dfrac{y'}{x'} = \dfrac{x}{x+2y}$, that is $mx + 2my = x$, or equivalently $y = \dfrac{1-m}{2m}\,x$. This is the line through $(0,0)$ with slope $\dfrac{1-m}{2m}$. Some isoclines are shown in yellow, the slope field is shown in green , and some solution curves are shown in blue.



(b) Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and draw the phase diagram.

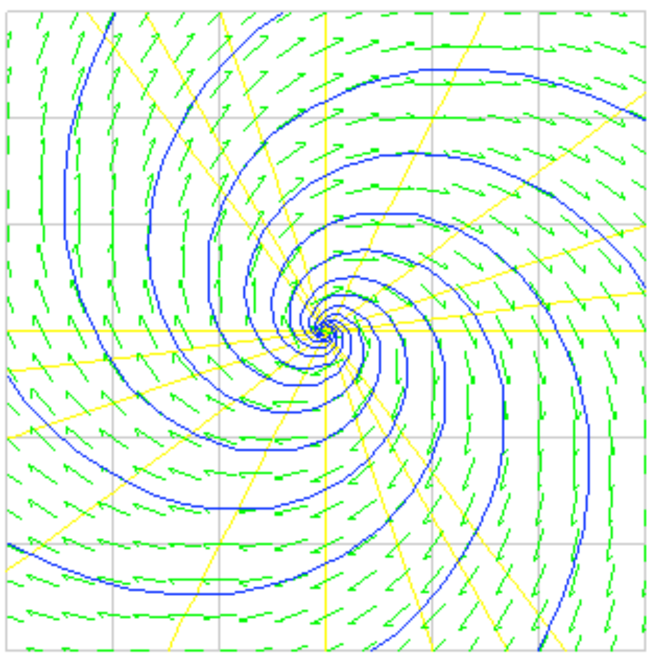
Solution: Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Then $|A - rI| = \begin{vmatrix} 1-r & 2 \\ -2 & 1-r \end{vmatrix} = r^2 - 2r + 5 = (r-1)^2 + 4$.
The eigenvalues are $r = 1 \pm 2i$.
When $r = 1 + 2i$ we have $A - rI = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \sim \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.
A complex solution is

$$\begin{pmatrix} z \\ w \end{pmatrix} = e^{(1+2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t\big(\cos 2t + i \sin 2t\big) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos 2t + i \sin 2t \\ -\sin 2t + i \cos 2t \end{pmatrix}.$$

Two independent real solutions are obtained from the real and imaginary parts of the above complex solution. The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^t \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + be^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

The isoclines are given by $m = \dfrac{y'}{x'} = \dfrac{-2x + y}{x + 2y}$, that is $mx + 2my = -2x + y$, or equivalently $y = \dfrac{2+m}{1-2m}\, x$.
This is the line through $(0, 0)$ of slope $\dfrac{2+m}{1-2m}$. Some isoclines are shown in yellow, the slope field is shown in green, and some solution curves are shown in blue.



**2:** (a) Find the solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$. Then $|A - rI| = \left| \begin{pmatrix} 1-r & -2 \\ 2 & -3-r \end{pmatrix} \right| = r^2 + 2r + 1 = (r+1)^2$.
When $r = -1$ we have $A - rI = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Note also that when $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have $(A - rI)\mathbf{v} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\mathbf{u}$.
Thus the general solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + bte^{-t} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

To get $x(0) = 2$ we need $2 = a + b$ and to get $y(0) = 1$ we need $1 = a$, so we must take $a = 1$ and $b = 1$, and we obtain the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + te^{-t} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

(b) Find the solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}$. Then $|A - rI| = \begin{vmatrix} 1-r & -2 \\ 4 & 5-r \end{vmatrix} = r^2 - 6r + 13 = (r-3)^2 + 4$,
The eigenvalues are $r = 3 \pm 2i$.
When $r = 3 + 2i$ we have $A - rI = \begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} \sim \begin{pmatrix} 2 & 1-i \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1-i \\ -2 \end{pmatrix}$.
A complex solution is

$$\begin{pmatrix} z \\ w \end{pmatrix} = e^{(3+2i)t} \begin{pmatrix} 1-i \\ -2 \end{pmatrix} = e^{3t} \big( \cos 2t + i \sin 2t \big) \begin{pmatrix} 1-i \\ -2 \end{pmatrix} = e^{3t} \begin{pmatrix} (\cos 2t + \sin 2t) + i(\sin 2t - \cos 2t) \\ -2\cos 2t - 2i \sin 2t \end{pmatrix}.$$

Two independent real solutions are given by the real and imaginary parts of this complex solution, so the general solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{3t} \begin{pmatrix} \sin 2t + \cos 2t \\ -2\cos 2t \end{pmatrix} + be^{3t} \begin{pmatrix} \sin 2t - \cos 2t \\ -2\sin 2t \end{pmatrix}.$$

To get $x(0) = 1$ we need $1 = a - b$ and to get $y(0) = 2$ we need $2 = -2a$, so we must have $a = -1$ and $b = -2$, and so the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = -e^{3t} \begin{pmatrix} \sin 2t + \cos 2t \\ -2\cos 2t \end{pmatrix} - 2e^{3t} \begin{pmatrix} \sin 2t - \cos 2t \\ -2\sin 2t \end{pmatrix} = e^{3t} \begin{pmatrix} \cos 2t - 3\sin 2t \\ 4\sin 2t + 2\cos 2t \end{pmatrix}.$$

**3:** (a) Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6t+1 \\ e^t \end{pmatrix}$.

Solution: First we solve the associated homogeneous system. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

Then $|A - rI| = \begin{vmatrix} 1-r & 2 \\ -1 & 4-r \end{vmatrix} = r^2 - 5r + 6 = (r-2)(r-3)$ so the eigenvalues are $r = 2, 3$.

When $r = 2$ we have $A - rI = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

When $r = 3$ we have $A - rI = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus the general solution to the homogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + be^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

We shall find a particular solution to the given non-homogeneous system using variation of parameters. We try $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = X \begin{pmatrix} p \\ q \end{pmatrix}$ where $X = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}$. Putting this in the non-homogeneous system gives

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}^{-1} \begin{pmatrix} 6t+1 \\ e^t \end{pmatrix} = e^{-5t} \begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{2t} & 2e^{2t} \end{pmatrix} \begin{pmatrix} 6t+1 \\ e^t \end{pmatrix}$$
$$= e^{-5t} \begin{pmatrix} (6t+1)e^{3t} - e^{4t} \\ -(6t+1)e^{2t} + 2e^{3t} \end{pmatrix} = \begin{pmatrix} (6t+1)e^{-2t} - e^{-t} \\ -(6t+1)e^{-3t} + 2e^{-2t} \end{pmatrix} .$$

We integrate (using integration by parts) to obtain

$$p = \int (6t+1)e^{-2t} - e^{-t}\, dt = -\tfrac{1}{2}(6t+1)e^{-2t} - \tfrac{3}{2}e^{-2t} + e^{-t} = -(3t+2)e^{-2t} + e^{-t} \text{ , and}$$
$$q = \int -(6t+1)e^{-3t} + 2e^{-2t}\, dt = \tfrac{1}{3}(6t+1)e^{-3t} + \tfrac{2}{3}e^{-3t} - e^{-t} = (2t+1)e^{-3t} - e^{-2t} .$$

Thus we obtain the particular solution

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} -(3t+2)e^{-2t} + e^{-t} \\ (2t+1)e^{-3t} - e^{-2t} \end{pmatrix} = \begin{pmatrix} -2(3t+2) + 2e^t + (2t+1) - e^t \\ -(3t+2) + e^t + (2t+1) - e^t \end{pmatrix} = \begin{pmatrix} -4t - 3 + e^t \\ -t - 1 \end{pmatrix} .$$

The general solution to the given system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + be^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4t - 3 + e^t \\ -t - 1 \end{pmatrix} .$$

(b) Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 5e^{3t} - 4 \end{pmatrix}$.

Solution: First we solve the associated homogeneous system. Let $A = \begin{pmatrix} 3 & -4 \\ 5 & -1 \end{pmatrix}$.

Then $\left|A - rI\right| = \begin{vmatrix} 3-r & -4 \\ 5 & -1-r \end{vmatrix} = r^2 - 2r + 17 = (r-1)^2 + 16$, so the eigenvalues are $r = 1 \pm 4i$. When $r = 1 + 4i$ we have $A - rI = \begin{pmatrix} 2-4i & -4 \\ 5 & -2-4i \end{pmatrix} \sim \begin{pmatrix} 1-2i & -2 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\mathbf{u} = \begin{pmatrix} 2 \\ 1-2i \end{pmatrix}$.

A complex solution is

$$\begin{pmatrix} z \\ w \end{pmatrix} = e^{(1+4i)t} \begin{pmatrix} 2 \\ 1-2i \end{pmatrix} = e^t \big(\cos 4t + i \sin 4t\big) \begin{pmatrix} 2 \\ 1-2i \end{pmatrix} = e^t \begin{pmatrix} 2\cos 4t + i \sin 4t \\ (2\sin 4t + \cos 4t) + i(\sin 4t - 2\cos 4t) \end{pmatrix}$$

so the general solution to the associated homogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^t \begin{pmatrix} 2\cos 4t \\ 2\sin 4t + \cos 4t \end{pmatrix} + be^t \begin{pmatrix} 2\sin 4t \\ \sin 4t - 2\cos 4t \end{pmatrix}.$$

We shall find a particular solution to the non-homogeneous system using the method of undetermined coefficients. We try $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + e^{3t} \begin{pmatrix} p \\ q \end{pmatrix}$. Put this into the system of DEs. The left side is

$$LS = \begin{pmatrix} {x_p}' \\ {y_p}' \end{pmatrix} = \begin{pmatrix} 3pe^{3t} \\ 3qe^{3t} \end{pmatrix}$$

and the right side is

$$\begin{aligned} RS &= \begin{pmatrix} 3 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x_p \\ y_p \end{pmatrix} + \begin{pmatrix} 1 \\ 5e^{3t} - 4 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} c + pe^{3t} \\ d + qe^{3t} \end{pmatrix} + \begin{pmatrix} 1 \\ 5e^{3t} - 4 \end{pmatrix} \\ &= \begin{pmatrix} 3c + 3pe^{3t} - 4d - 4qe^{3t} + 1 \\ 5c + 5pe^{3t} - d - qe^{3t} + 5e^{3t} - 4 \end{pmatrix}. \end{aligned}$$

In order to have $LS = RS$ we need $3pe^{3t} = 3c + 3pe^{3t} - 4d - 4qe^{3t} + 1 = 0$, that is $(3c - 4d + 1) + (-4q)e^{3t} = 0$, and $3qe^{3t} = 5c + 5pe^{3t} - d - qe^{3t} + 5e^{3t} - 4$, that is $(5c - d - 4) + (5p - 4q + 5)e^{3t} = 0$. Since 1 and $e^{3t}$ are linearly independent, the coefficients must all vanish, so we have $3c - 4d + 1 = 0$, $5c - d - 4 = 0$, $-4q = 0$ and $5p - 4q + 5 = 0$. The first two of these equations give $c = 1$ and $d = 1$ and the second two give $p = -1$ and $q = 0$. Thus we obtain the particular solution

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The general solution to the given system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^t \begin{pmatrix} 2\cos 4t \\ 2\sin 4t + \cos 4t \end{pmatrix} + be^t \begin{pmatrix} 2\sin 4t \\ \sin 4t - 2\cos 4t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

**4:** Find the solution to the system $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 2 & -4 & 3 \end{pmatrix}$. Then

$$\begin{aligned} \big|A - rI\big| &= \begin{vmatrix} 2-r & -2 & 1 \\ 1 & -1-r & 1 \\ 2 & -4 & 3-r \end{vmatrix} \\ &= -(r-2)(r-3)(r+1) - 4 - 4 - 4(r-2) - 2(r-3) + 2(r+1) \\ &= -(r-2)(r-3)(r+1) - 4r + 8 = -(r-2)\big((r-3)(r+1)+4\big) \\ &= -(r-2)(r^2-2r+1) = -(r-2)(r-1)^2 \,. \end{aligned}$$

When $r = 2$ we have

$$A - rI = \begin{pmatrix} 0 & -2 & 1 \\ 1 & -3 & 1 \\ 2 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. When $r = 1$ we have

$$A - rI = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we have the two independent eigenvectors $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Thus the general solution to the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ae^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + be^t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + ce^t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} .$$

To get $x(0) = 1$ we need $a + b + 2c = 1$, to get $y(0) = 2$ we need $a + 0b + c = 2$, and to get $z(0) = 1$ we need $2a - b + 0c = 1$. We solve these three equations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{array}\right) .$$

Thus $a = -2$, $b = -5$ and $c = 4$, so the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 5e^t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 4e^t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + e^t \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} .$$

**5:** Solve the system $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ -2 & -2 & 2 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + e^{-2t} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 1 & 1 & -2 \\ -2 & -2 & 2 \\ 3 & 2 & -3 \end{pmatrix}$. Then

$$\begin{aligned} \left|A - rI\right| &= \begin{vmatrix} 1-r & 1 & -2 \\ -2 & -2-r & 2 \\ 3 & 2 & -3-r \end{vmatrix} \\ &= -(r-1)(r+2)(r+3) + 6 + 8 + 4(r-1) - 2(r+3) - 6(r+2) \\ &= -(r-1)(r+2)(r+3) - 4r - 8 = -(r+2)\big((r-1)(r+3)+4\big) \\ &= -(r+2)(r^2+2r+1) = -(r+2)(r+1)^2 . \end{aligned}$$

When $r = -2$ we have

$$A - rI = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 0 & 2 \\ 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 2 \\ 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so an eigenvector is $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. When $r = -1$ we have

$$A - rI = \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \\ 3 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 2 \\ 3 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

so one eigenvector is $\mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. We look for a vector $\mathbf{w}$ such that $(A - rI)\mathbf{w} = \mathbf{v}$:

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 2 \\ -2 & -1 & 2 & -2 \\ 3 & 2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -2 & -1 & 2 & -2 \\ 3 & 2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 1 & 2 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and we obtain $\mathbf{w} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$. Thus the general solution to the associated homogeneous system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ae^{-2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + be^{-t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + ce^{-t} \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + cte^{-t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} .$$

To find a particular solution to the given non-homogeneous system, we use the method of undetermined coefficients. We try $\mathbf{x} = e^{-2t}\mathbf{p} + te^{-2t}\mathbf{q}$, where $\mathbf{x} = \big(x_p, y_p, z_p\big)^T$ and $\mathbf{p}$ and $\mathbf{q}$ are vectors in $\mathbf{R}^3$. We put $\mathbf{x}$ into the non-homogeneous system of DEs. The left side is

$$LS = \mathbf{x}' = -2e^{-2t}\mathbf{p} + e^{-2t}\mathbf{q} - 2te^{-2t}\mathbf{q} = e^{-2t}(-2\mathbf{p} + \mathbf{q}) + te^{-2t}(-2\mathbf{q})$$

and, writing $\mathbf{b} = \big(-1, 2, 2\big)^T$, the right side is

$$RS = A\mathbf{x} + e^{-2t}\mathbf{b} = A\big(e^{-2t}\mathbf{p} + te^{-2t}\mathbf{q}\big) + e^{-2t}\mathbf{b} = e^{-2t}(A\mathbf{p} + \mathbf{b}) + te^{-2t}A\mathbf{q}\,.$$

To get $LS = RS$ we need $-2\mathbf{p} + \mathbf{q} = A\mathbf{p} + \mathbf{b}$ and $-2\mathbf{q} = A\mathbf{q}$. Note that $-2\mathbf{q} = A\mathbf{q} \iff (A + 2I)\mathbf{q} = 0$ so $\mathbf{q}$ must be an eigenvector of $r = -2$, and so we must have $\mathbf{q} = k\mathbf{u}$ for some $k \in \mathbf{R}$. Also, note that $-2\mathbf{p} + \mathbf{q} = A\mathbf{p} + \mathbf{b} \iff (A + 2I)\mathbf{p} = \mathbf{q} - \mathbf{b}$, so we solve this to find $\mathbf{p}$:

$$\left(\begin{array}{ccc|c} 3 & 1 & -2 & k+1 \\ -2 & 0 & 2 & -k-2 \\ 3 & 2 & -1 & k-2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -2 & 0 & 2 & -k-2 \\ 3 & 2 & -1 & k-2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 2 & 2 & -k-4 \\ 0 & 1 & 1 & -k-1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -k-1 \\ 0 & 2 & 2 & -k-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & k \\ 0 & 1 & 1 & -k-1 \\ 0 & 0 & 0 & k-2 \end{array}\right)$$

To get a solution, we must take $k = 2$ and we have $\mathbf{q} = k\mathbf{u} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, and $\mathbf{p} = \begin{pmatrix} k \\ -k-1 \\ k-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$.

Thus we obtain the particular solution

$$\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = e^{-2t}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + te^{-2t}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

The general solution to the given (non-hommogeneous) system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ae^{-2t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + be^{-t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + ce^{-t}\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + cte^{-t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + te^{-2t}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$