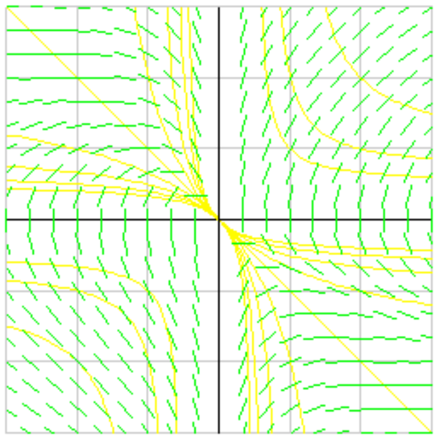
MATH 218 Differential Equations, Solutions to Assignment 10

1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$.

(a) Sketch the isoclines $m = 0, \pm\frac{1}{2}, \pm 1 \pm 2$ and sketch the slope field for this system.

Solution: The isoclines are given by $m = \dfrac{y'}{x'} = \dfrac{x+y}{xy}$, that is $mxy = x+y$ or equivalently $y = \dfrac{x}{mx-1}$. This is the hyperbola through the point $(0,0)$ with vertical asymptote along $x = \frac{1}{m}$ and horizontal asymptote along $y = \frac{1}{m}$. The isoclines $m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$ are shown in yellow, and the slope field is shown in green.



(b) Use Euler's method with step size $\Delta t = \frac{1}{2}$ to approximate the point $\big(x(2), y(2)\big)$, where $\big(x(t), y(t)\big)$ is the solution to the above system with $\big(x(0), y(0)\big) = (-1, 1)$.

Solution: We let $t_0 = 0$, $x_0 = -1$, $y_0 = 1$, then set $t_{k+1} = t_k + \Delta t$, $x_{k+1} = x_k + (x_k y_k)\Delta t$ and $y_{k+1} = y_k = (x_k + y_k)\Delta t$. The first few values of t_k, x_k, y_k, $x_k y_k$ and $(x_k + y_k)$ are shown in the table below.

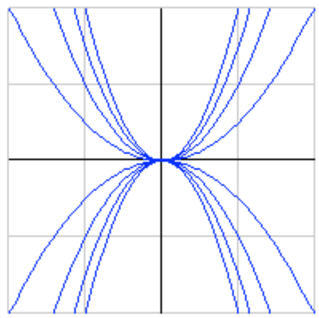
k	t_k	x_k	y_k	$x_k y_k$	$x_k + y_k$
0	0	-1	1	-1	0
1	$\frac{1}{2}$	$-\frac{3}{2}$	1	$-\frac{3}{2}$	$-\frac{1}{2}$
2	1	$-\frac{9}{4}$	$\frac{3}{4}$	$-\frac{27}{16}$	$-\frac{3}{2}$
3	$\frac{3}{2}$	$-\frac{99}{32}$	0	0	$-\frac{99}{32}$
4	2	$-\frac{99}{32}$	$-\frac{99}{64}$		

Thus we have $\big(x(2), y(2)\big) \cong (x_4, y_4) = \big(-\frac{99}{32}, -\frac{99}{64}\big)$.

2: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{y} \\ \frac{2}{x} \end{pmatrix}$.

(a) Sketch the phase diagram for this system.

Solution: We have $\frac{dy}{dx} = \frac{2y}{x}$. This DE is separable since we can write it as $\frac{y'}{y} = \frac{2}{x}$. Integrate both sides to get $\ln|y| = 2\ln|x| + a$, or equivalently, $y = bx^2$. This tells us that the solution curves lie on the parabolas $y = bx^2$, (so we can sketch the phase diagram without sketching the slope field). Also note that in the upper-half plane (where $y > 0$) we have $x' > 0$ so $\big(x(t), y(t)\big)$ is moving from left to right, while in the lower-half plane it is moving from right to left. Arrows to indicate the direction of motion should be added to the following picture.



(b) Solve the system by eliminating y and y' from x'' to get a second order DE for $x = x(t)$.

Solution: We have two DEs, $x' = \frac{1}{y}$ (1) and $y' = \frac{2}{x}$ (2). Differentiate equation (1), then use equations (1) and (2) to get $x'' = -\frac{1}{y^2} \cdot y' = -(x')^2 \cdot \frac{2}{x}$, that is $x\,x'' = 2(x')^2$. Since this second order DE does not involve the variable t, we let $x' = u$ and $x'' = u\,u'$, and the DE becomes $xuu' + 2u^2 = 0$, that is $u' + \frac{2}{x}\,u = 0$. This is linear. An integrating factor is $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and the solution is $u = \frac{1}{x^2}\int 0\,dx = \frac{a}{x^2}$. Replace u by x' again, and we have the DE $x' = \frac{a}{x^2}$, that is $x^2\,x' = a$. Integrate both sides to get $\frac{1}{3}x^3 = at + b$, that is $x = \big(3at + 3b\big)^{1/3}$. Rewrite this as $x = \big(pt + q\big)^{1/3}$. Note that $x' = \frac{p}{3}\big(pt + q\big)^{-2/3}$. From equation (1) we have $y = \frac{1}{x'} = \frac{3}{p}\big(pt + q\big)^{2/3}$. Thus the solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(pt+q\big)^{1/3} \\ \frac{3}{p}\big(pt+q\big)^{2/3} \end{pmatrix}.$$

(c) In particular, find the solution to the system with $\big(x(0), y(0)\big) = (2, 1)$.

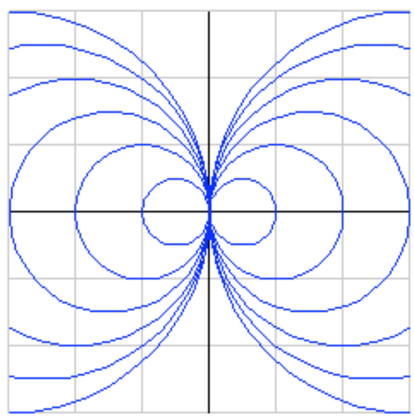
Solution: Put $t = 0$, $x = 2$ and $y = 1$ into our solutions $x = \big(pt + q\big)^{1/3}$ and $y = \frac{1}{x'} = \frac{3}{p}\big(pt + q\big)^{2/3}$ to get $2 = q^{1/3}$ and $1 = \frac{3}{p}\,q^{2/3}$, so we must have $q = 8$ and $p = 12$. Thus the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(12t+8\big)^{1/3} \\ \frac{1}{4}\big(12t+8\big)^{2/3} \end{pmatrix}.$$

3: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy \\ \frac{1}{2}(y^2 - x^2) \end{pmatrix}$.

(a) Sketch the phase diagram for this system.

Solution: We have $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{y^2 - x^2}{2xy}$. This DE is homogeneous since we can write it as $\dfrac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$. We make the substitution $u = \frac{y}{x}$ and the DE becomes $u + xu' = \frac{1}{2}\left(u - \frac{1}{u}\right)$, that is $xu' = -\frac{1}{2}\left(u + \frac{1}{u}\right)$. This is separable since we can rewrite it as $\dfrac{2u\,u'}{u^2+1} = -\dfrac{1}{x}$. Integrate both sides to get $\ln(u^2+1) = -\ln|x| + b$, or equivalently $u^2 + 1 = \frac{a}{x}$. Replace u by $\frac{y}{x}$ again, and we get $\frac{y^2}{x^2} + 1 = \frac{a}{x}$, that is $x^2 - ax + y^2 = 0$. Complete the square to write this as $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$. This is the circle centered at $\left(\frac{a}{2}, 0\right)$ of radius $\frac{a}{2}$. The solution curves all lie on these circles. Also, since $x' = xy$, we see that $\big(x(t), y(t)\big)$ moves from left to right in the 1st and 3rd quadrants, and it moves from right to left in the 2nd and 4th quadrants. Arrows to indicate the direction of motion should be added to the following phase diagram. (There are also two solution curves that follow the positive and negative y-axis).



(b) Find the solution $\big(x(t), y(t)\big)$ to the above system with $\big(x(-1), y(-1)\big) = (1, 1)$ by first finding the function $f(x)$ such that the solution curve lies on the graph $y = f(x)$.

Solution: From part (a) we know the solution follows one of the circles $x^2 - ax + y^2 = 0$. Put in $t = 0$, $x = 1$ and $y = 1$ to get $1 - a + 1 = 0$ so $a = 2$, so the solution curve lies on the circle $x^2 - 2x + y^2 = 0$, that is $y = \sqrt{2x - x^2}$ (we use the positive square root since $y(-1) = 1 > 0$). Put this into the DE $x' = xy$ to get $x' = x\sqrt{2x - x^2}$. This is separable since we can write it as $\dfrac{x'}{x\sqrt{2x - x^2}} = 1$. Integrate both sides (with respect to t) to get $\displaystyle\int \frac{dx}{x\sqrt{2x - x^2}} = \int dt = t + c$. To solve the integral on the left, make the substitution $\sin\theta = x - 1$, $\cos\theta = \sqrt{2x - x^2}$, $\cos\theta\, d\theta$ to get

$$\int \frac{dx}{x\sqrt{2x - x^2}} = \int \frac{\cos\theta\, d\theta}{(1 + \sin\theta)\cos\theta} = \int \frac{d\theta}{1 + \sin\theta} = \int \frac{1 - \sin\theta}{\cos^2\theta}\, d\theta = \int \sec^\theta - \sec\theta\tan\theta\; d\theta$$

$$= \tan\theta - \sec\theta = \frac{x - 1}{\sqrt{2x - x^2}} - \frac{1}{\sqrt{2x - x^2}} = \frac{x - 2}{\sqrt{2x - x^2}} = -\sqrt{\frac{2 - x}{x}}\,.$$

Thus the solution to the DE is given by $-\sqrt{\dfrac{2 - x}{x}} = t + c$. Put in $t = -1$ and $x = 1$ to get $-1 = -1 + c$ so $c = 0$, and the solution is given by $-\sqrt{\dfrac{2 - x}{x}} = t$. Square both sides to get $\dfrac{2 - x}{x} = t^2$, so $2 - x = xt^2$, that is $x = \frac{2}{t^2 + 1}$. Finally, use the DE $x' = xy$ to get $y = \dfrac{x'}{x} = \dfrac{-4t}{(t^2+1)^2} \cdot \dfrac{t^2 + 1}{2} = \dfrac{-2t}{t^2 + 1}$. Thus the solution is $x(t) = \dfrac{2}{t^2 + 1}$ and $y(t) = \dfrac{-2t}{t^2 + 1}$.

4: Use the method of reduction of order to solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{t} \\ -\frac{1}{t} \end{pmatrix}$ is one solution.

Solution: We try $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \left(1+\frac{1}{t}\right) \\ 0 & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \left(1+\frac{1}{t}\right) v \\ -\frac{1}{t}\, v \end{pmatrix}$. We put this into the DE. The left side is

$$LS = \begin{pmatrix} {x_2}' \\ {y_2}' \end{pmatrix} = \begin{pmatrix} u' - \frac{1}{t^2}\, v + \left(1+\frac{1}{t}\right) v' \\ \frac{1}{t^2}\, v - \frac{1}{t}\, v' \end{pmatrix}$$

and the right side is

$$RS = \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 1 \end{pmatrix} \begin{pmatrix} u + \left(1+\frac{1}{t}\right) v \\ -\frac{1}{t}\, v \end{pmatrix} = \begin{pmatrix} -\frac{1}{t^2}\, v \\ \frac{1}{t}\, u + \frac{1}{t^2}\, v \end{pmatrix}$$

To get $LS = RS$ we need $u' + \left(1+\frac{1}{t}\right) v' = 0$ (1) and $-\frac{1}{t}\, v' = \frac{1}{t}\, u$ (2). Equation (2) gives $v' = -u$, and we put this into equation (1) to get $u' - \left(1+\frac{1}{t}\right) u = 0$. This is linear. An integrating factor is $\lambda = e^{\int -\left(1+\frac{1}{t}\right) dt} = e^{-t-\ln t} = \dfrac{1}{te^t}$, and the solution is $u = te^t \displaystyle\int 0\, dt = ate^t$. We choose $a = 1$ so that $u = te^t$. Since $v' = -u$ we have $v' = -te^t$ so $v = \displaystyle\int -te^t\, dt$. Integrate by parts to get $v = \displaystyle\int -te^t\, dt = -te^t + \int e^t\, dt = -te^t + e^t + b$. We choose $b = 0$ so that $v = (1-t)e^t$. Thus we obtain a second solution to the system

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} u + \left(1+\frac{1}{t}\right) v \\ -\frac{1}{t}\, v \end{pmatrix} = \begin{pmatrix} te^t + \left(1+\frac{1}{t}\right)(1-t)e^t \\ -\frac{1}{t}(1-t)e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{t}\, e^t \\ \left(1-\frac{1}{t}\right) e^t \end{pmatrix},$$

and the general solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A \begin{pmatrix} 1+\frac{1}{t} \\ -\frac{1}{t} \end{pmatrix} + Be^t \begin{pmatrix} \frac{1}{t} \\ 1-\frac{1}{t} \end{pmatrix}.$$

5: Use reduction of order and variation of parameters to solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t^2} \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix}$ given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} \\ -1 \end{pmatrix}$ is one solution to the associated homogeneous system.

Solution: First we use reduction of order to find a second independent solution to the homogeneous system. We try

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + x_1 v \\ y_1 v \end{pmatrix} = \begin{pmatrix} u + \frac{1}{t}\, v \\ -v \end{pmatrix}.$$

We put this into the associated homogeneous DE. The left side is

$$LS = \begin{pmatrix} {x_2}' \\ {y_2}' \end{pmatrix} = \begin{pmatrix} u' - \frac{1}{t^2}\, v + \frac{1}{t}\, v' \\ -v' \end{pmatrix}$$

and the right side is

$$RS = \begin{pmatrix} 0 & \frac{1}{t^2} \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t^2} \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} u + \frac{1}{t}\, v \\ -v \end{pmatrix} = \begin{pmatrix} -\frac{1}{t^2}\, v \\ u \end{pmatrix}.$$

To get $LS = RS$ we need $u' + \frac{1}{t}\, v' = 0$ (1) and $-v' = u$ (2). From (2) we have $v' = -u$. We put this in (1) to get $u' - \frac{1}{t}\, u = 0$. This is linear. An integrating factor is $\lambda = e^{\int -\frac{1}{t}\, dt} = e^{-\ln t} = \frac{1}{t}$ and the solution is $u = t \displaystyle\int 0\, dt = at$. Then we have $v' = -u = -at$ so $v = \displaystyle\int -at\, dt = -\frac{1}{2}at^2 + b$. We choose $a = 2$ and $b = 0$ so that $u = 2t$ and $v = -t^2$. Thus we obtain the second solution

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} u + \frac{1}{t}\, v \\ -v \end{pmatrix} = \begin{pmatrix} 2t - t \\ t^2 \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Now that we have two independent solutions to the associated homogeneous system, we use variation of parameters to find a particular solution to the given non-homogeneous system. We try $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = X \begin{pmatrix} u \\ v \end{pmatrix}$ where $X = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix}$. Putting this into the given system gives

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = X^{-1} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \frac{1}{2t} \begin{pmatrix} t^2 & -t \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \frac{1}{2t} \begin{pmatrix} 4t^2 - 3t\sqrt{t} \\ 4 + \frac{3}{\sqrt{t}} \end{pmatrix}.$$

Since $u' = 2t - \frac{3}{2}\, t^{1/2}$ we have $u = \displaystyle\int 2t - \frac{3}{2}\, t^{1/2}\, dt = t^2 - t^{3/2}$ (plus a constant which we choose to be zero), and since $v' = 2t^{-1} + \frac{3}{2}\, t^{-3/2}$ we have $v = 2\ln t - 3t^{-1/2}$ (plus a constant). Thus we obtain the particular solution

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix} \begin{pmatrix} t^2 - t^{3/2} \\ 2\ln t - 3t^{-1/2} \end{pmatrix} = \begin{pmatrix} t - t^{1/2} + 2t\ln t - 3t^{1/2} \\ -t^2 + t^{3/2} + 2t^2 \ln t - 3t^{3/2} \end{pmatrix}.$$

The general solution to the given (non-homogeneous) system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} = A \begin{pmatrix} \frac{1}{t} \\ -1 \end{pmatrix} + B \begin{pmatrix} t \\ t^2 \end{pmatrix} + \begin{pmatrix} t + 2t\ln t - 4t^{1/2} \\ -t^2 + 2t^2 \ln t - 2t^{3/2} \end{pmatrix}.$$