# MATH 148 Calculus 2, Solutions to the Exercises for Chapter 7

**1:** (a) Define $f_n:[0,\infty)\to\mathbb{R}$ by $f_n(x)=nxe^{-nx}$. Find the pointwise limit $f(x)=\lim\limits_{n\to\infty}f_n(x)$ and determine whether $f_n\to f$ uniformly on $[0,\infty)$.

Solution: Note that $f_n(0)=0$ hence $\lim\limits_{n\to\infty}f_n(0)=0$. When $x>0$, we have $\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}\dfrac{nx}{e^{nx}}=0$ by l'Hôpitals' Rule, indeed

$$\lim_{n\to\infty}\frac{nx}{e^{nx}}=\lim_{r\to\infty}\frac{rx}{e^{rx}}=\lim_{r\to\infty}\frac{\frac{d}{dr}(rx)}{\frac{d}{dr}(e^{rx})}=\lim_{r\to\infty}\frac{x}{xe^{rx}}=\lim_{r\to\infty}\frac{1}{e^{rx}}=0$$

since $e^{rx}\to\infty$ as $r\to\infty$. Thus the pointwise limit is $f(x)=\lim\limits_{n\to\infty}nxe^{-nx}=0$ for all $x\in[0,\infty)$. In other words we have $f_n\to 0$ pointwise on $[0,\infty)$.

Note that $f_n\big(\frac{1}{n}\big)=\frac{1}{e}$ for all $n\in\mathbb{Z}^+$, and so the convergence is not uniform. To be very explicit, $f_n\to 0$ uniformly on $[0,\infty)$ means that $\forall\,\epsilon>0\ \exists\,m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\ \forall x\in[0,\infty)\ \big(n\geq m\implies\big|f_n(x)-0\big|<\epsilon\big)$, so the convergence is not uniform when $\exists\,\epsilon>0\ \forall m\in\mathbb{Z}^+\ \exists n\in\mathbb{Z}^+\ \exists x\in[0,\infty)\ \big(n\geq m$ and $\big|f_n(x)-0\big|\geq\epsilon\big)$. To prove this, we choose $\epsilon=\frac{1}{e}$, we let $m\in\mathbb{Z}^+$, we choose $n=m$, and we choose $x=\frac{1}{n}$, and then we have $n\geq m$ and $\big|f_n(x)-0\big|=f_n\big(\frac{1}{n}\big)=\frac{1}{e}\geq\epsilon$.

(b) Define $f_n:[0,\infty)\to\mathbb{R}$ by $f_n(x)=\frac{x}{1+nx^2}$. Find the pointwise limit $f(x)=\lim\limits_{n\to\infty}f_n(x)$ and determine whether $f_n\to f$ uniformly on $[0,\infty)$.

Solution: Note that $\lim\limits_{n\to\infty}f_n(0)=\lim\limits_{n\to\infty}0=0$, and when $x>0$ we have $\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}\frac{x}{1+nx^2}=0$ since $1+nx^2\to\infty$ as $n\to\infty$. Thus the pointwise limit is $f(x)=\lim\limits_{n\to\infty}\frac{x}{1+nx^2}=0$ for all $x\in[0,\infty)$. In other words, we have $f_n\to 0$ pointwise on $[0,\infty)$.

Let $n\in\mathbb{Z}^+$. Note that $f_n(x)=\frac{x}{1+nx^2}\geq 0$ for all $x\in[0,\infty)$ and we have ${f_n}'(x)=\frac{(1+nx^2)-x(2nx)}{(1+nx^2)^2}=\frac{1-nx^2}{(1+nx^2)^2}$ so that ${f_n}'(x)>0$ when $0\leq x<\frac{1}{\sqrt{n}}$ and ${f_n}'(x)<0$ when $x>\frac{1}{\sqrt{n}}$. By the First Derivative Test, $f_n(x)$ attains its maximum value at $x=\frac{1}{\sqrt{n}}$ and the maximum value is $f_n\big(\frac{1}{\sqrt{n}}\big)=\frac{1}{2\sqrt{n}}$. Since $\big|f_n(x)\big|=f_n(x)\leq\frac{1}{2\sqrt{n}}$ for all $x\in[0,\infty)$ and $\frac{1}{2\sqrt{n}}\to 0$ as $n\to\infty$, it follows that $f_n\to 0$ uniformly on $[0,\infty)$. To be very explicit, let $\epsilon>0$, choose $m\in\mathbb{Z}^+$ so that $\frac{1}{2\sqrt{m}}<\epsilon$, let $n\in\mathbb{Z}^+$ and let $x\in[0,\infty)$. Suppose that $n\geq m$. Then we have $\big|f_n(x)-0\big|=f_n(x)\leq\frac{1}{2\sqrt{n}}\leq\frac{1}{2\sqrt{m}}<\epsilon$.

(c) Define $f_n:[0,\infty]\to\mathbb{R}$ by $f_n(x)=\frac{x+n}{x+4n}$. Show that $(f_n)$ converges uniformly on $[0,r]$ for every $r>0$ but that $(f_n)$ does not converge uniformly on $[0,\infty)$.

Solution: The pointwise limit is $f(x)=\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}\frac{x+n}{x+4n}=\frac{1}{4}$, so we have $f_n\to\frac{1}{4}$ pointwise on $[0,\infty)$. Note that ${f_n}'(x)=\frac{(x+4n)-(x+n)}{(x+4n)^2}=\frac{3n}{(x+4n)^2}>0$ for all $x\in[0,\infty)$ so that $f_n(x)$ is strictly increasing on $[0,\infty)$ with $f_n(0)=\frac{1}{4}$ and $\lim\limits_{x\to\infty}f_n(x)=\lim\limits_{x\to\infty}\frac{x+n}{x+4n}=1$. Because $\lim\limits_{x\to\infty}f_n(x)=1$ it follows that $(f_n)$ does not converge uniformly on $[0,\infty)$ to the constant function $\frac{1}{4}$. To be explicit, choose $\epsilon=\frac{1}{2}$, let $m\in\mathbb{Z}^+$, choose $n\geq m$, and choose $x\in[0,\infty)$ large enough so that $\big|f_n(x)-1\big|\leq\frac{1}{4}$. Then we have $f_n(x)\geq 1-\frac{1}{4}=\frac{3}{4}$ so that $\big|f_n(x)-\frac{1}{4}\big|\geq\frac{3}{4}-\frac{1}{4}=\frac{1}{4}=\epsilon$.

On the other hand, we claim that for every $r>0$ we have $f_n\to\frac{1}{4}$ uniformly on $[0,r]$. Let $r>0$. Note that $f_n(0)=\frac{1}{4}$ and for $0<x\leq r$ we have

$$\Big|f_n(x)-\frac{1}{4}\Big|=\Big|\frac{x+n}{x+4n}-\frac{1}{4}\Big|=\frac{3x}{4(x+4n)}=\frac{3}{4+\frac{16n}{x}}\leq\frac{3}{4+\frac{16n}{r}}\longrightarrow 0\text{ as }n\to\infty.$$

It follows that $f_n\to\frac{1}{4}$ uniformly on $[0,r]$, as claimed. Indeed, to be explicit, let $\epsilon>0$, choose $m\in\mathbb{Z}^+$ large enough that $\frac{3}{4+\frac{16m}{r}}<\epsilon$, let $x\in[0,r]$ and let $n\in\mathbb{Z}^+$ with $n\geq m$. Then $\big|f_n(x)-\frac{1}{4}\big|\leq\frac{3}{4+\frac{16n}{r}}\leq\frac{3}{4+\frac{16m}{r}}<\epsilon$.

**2:** (a) Find $\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx$ and $\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx$.

Solution: Let $x\in[0,1]$. If $x=0$ or $x=1$ then $nx(1-x^2)^n=0$ for all $n$ and so $\lim_{n\to\infty} nx(1-x^2)^n=0$. If $x\in(0,1)$ then $0<(1-x^2)<1$, so the series $\sum nx(1-x^2)^n$ converges by the Ratio Test and so $\lim_{n\to\infty} nx(1-x^2)^n=0$ by the Divergence Test. Thus $\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx=\int_0^1 0\,dx=0$. On the other hand, using the substitution $u=1-x^2$ so $du=-2x\,dx$ we have

$$\int_0^1 nx(1-x^2)^n\,dx=\int_1^0 -\tfrac{1}{2}n\,u^2\,du=\left[\frac{-n\,u^{n+1}}{2(n+1)}\right]_1^0=\frac{n}{2(n+1)}\,,$$

and so we have $\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx=\frac{1}{2}$.

(b) Find $\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx$ and $\lim_{n\to\infty}\int_1^4 \frac{\tan^{-1}(nx)}{x}\,dx$.

Solution: Let $x\in[1,4]$. Then $\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}=\frac{\pi}{2x}$ and so

$$\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\frac{\pi}{2x}\,dx=\Big[\tfrac{\pi}{2}\ln x\Big]_1^4=\pi\ln 2\,.$$

We claim that $\left\{\frac{\tan^{-1}(nx)}{x}\right\}\to\frac{\pi}{2x}$ uniformly on $[1,4]$. Indeed, given $\epsilon>0$ we can choose $N$ so that $x\geq N\implies \left|\tan^{-1}x-\frac{\pi}{2}\right|<\epsilon$ for all $x\geq N$. Then for $n\geq N$ and $x\geq 1$ we have

$$\left|\frac{\tan^{-1}(nx)}{x}-\frac{\pi}{2x}\right|=\frac{\left|\tan^{-1}(nx)-\frac{\pi}{2}\right|}{x}<\frac{\epsilon}{x}\leq\epsilon\,.$$

Since the convergence is uniform, $\lim_{n\to\infty}\int_1^4\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\pi\ln 2$.

(c) Show that $\sum_{n=0}^{\infty}\frac{\cos(2^n x)}{1+n^2}$ converges uniformly on $\mathbb{R}$ and find $\int_0^{\pi/4}\sum_{n=0}^{\infty}\frac{\cos(2^n x)}{1+n^2}\,dx$.

Solution: For all $x\in\mathbb{R}$ we have $\left|\frac{\cos(2^n x)}{1+n^2}\right|\leq\frac{1}{1+n^2}<\frac{1}{n^2}$, and $\sum\frac{1}{n^2}$ converges, so $\sum_{n=0}^{\infty}\frac{\cos(2^n x)}{1+n^2}$ converges uniformly by the Weirstrass M-Test. Since the convergence is uniform,

$$\int_0^{\pi/4}\sum_{n=0}^{\infty}\frac{\cos(2^n x)}{1+n^2}\,dx=\sum_{n=0}^{\infty}\int_0^{\pi/4}\frac{\cos(2^n x)}{1+n^2}\,dx=\sum_{n=0}^{\infty}\left[\frac{1}{2^n}\,\frac{\sin(2^n x)}{1+n^2}\right]_0^{\pi/4}=\tfrac{\sqrt{2}}{2}+\tfrac{1}{4}+0+0+\cdots=\tfrac{\sqrt{2}}{2}+\tfrac{1}{4}.$$

(d) Show that $\sum_{n=1}^{\infty}\sin\left(\frac{x}{n^2}\right)$ converges uniformly on any closed interval $[a,b]$.

Solution: Note that $|\sin x|\leq|x|$ for all $x\in\mathbb{R}$ and so $\left|\sin\left(\frac{x}{n^2}\right)\right|\leq\frac{|x|}{n^2}$ for all $x$. Let $[a,b]$ be any closed interval and let $M=\max(|a|,|b|)$. Then for $x\in[a,b]$ we have $|x|\leq M$ and so $\left|\sin\left(\frac{x}{n^2}\right)\right|\leq\frac{|x|}{n^2}\leq\frac{M}{n^2}$. Since $\sum\frac{M}{n^2}$ converges, $\sum\sin\left(\frac{x}{n^2}\right)$ converges uniformly on $[a,b]$ by the Weirstrass M-Test.

**3:** Determine which of the following statements are true for all sequences of functions $(f_n)$ and $(g_n)$ and all $E \subseteq \mathbb{R}$.

(a) If $(f_n)$ and $(g_n)$ converge uniformly on $E$ then $(f_n g_n)$ converge uniformly on $E$.

Solution: This is FALSE. Let $E = \mathbb{R}$, let $f(x) = g(x) = x$ and let $f_n(x) = g_n(x) = x + \frac{1}{n}$. Then we have $f_n(x)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ so $\lim\limits_{n\to\infty} f_n(x)^2 = x^2 = f(x)^2$ for all $x \in \mathbb{R}$, but the convergence is not uniform, since given any positive integer $n$, when $x \geq n$ we have $|f_n(x)^2 - f(x)^2| = \frac{2x}{n} + \frac{1}{n^2} > 2$.

(b) Show that if $(f_n)$ and $(g_n)$ converge uniformly on $E$ and $f$ and $g$ are bounded on $E$ then $(f_n g_n)$ converges uniformly on $E$.

Solution: This is TRUE. Suppose that $(f_n)$ and $(g_n)$ converge uniformly on $E$ and $f$ and $g$ are bounded on $E$, say $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in E$. Choose $N_1$ so that $n \geq N_1 \Longrightarrow |f_n(x) - f(x)| < 1$. Note that for $n \geq N_1$ we have $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq M + 1$. Now choose $N \geq N_1$ so that when $n \geq N$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2(M+1)}$ for all $x$. Then when $n \geq N$ we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq (M+1)\tfrac{\epsilon}{2(M+1)} + \tfrac{\epsilon}{2M}\, M = \epsilon\,. \end{aligned}$$

Thus $f_n g_n \to fg$ uniformly on $E$.

(c) If $(f_n)$ converges uniformly on $(a,b)$ and pointwise on $[a,b]$ then $(f_n)$ converges uniformly on $[a,b]$.

Solution: This is TRUE. Indeed, suppose that $(f_n)$ converges uniformly in $(a,b)$ and that $\big(f_n(a)\big)$ and $\big(f_n(b)\big)$ both converge. Then given $\epsilon > 0$ we can choose $N$ so that when $l, m \geq N$ we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in (0,1)$, and $|f_l(a) - f_m(a)| < \epsilon$ and $|f_l(b) = f_m(b)| < \epsilon$, and so we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in [a,b]$.

(d) If each $f_n$ is continuous on $[a,b]$ and $\sum f_n$ converges uniformly on $[a,b]$ then $\sum M_n$ converges, where $M_n = \max\big\{|f_n(x)|\big|a \leq x \leq b\big\}$.

Solution: This is FALSE. For a counterexample, let

$$f_n(x) = \begin{cases} \frac{1}{n}\sin^2(2^n \pi\, x)\ , \text{ if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \\ 0\ , \text{ otherwise.} \end{cases}$$

Then $M_n = \frac{1}{n}$ so $\sum M_n$ diverges, and yet we claim that $\sum f_n$ converges uniformly on $[0,1]$. Indeed if we write $S(x) = \sum\limits_{n=1}^{\infty} f_n(x)$ and $S_l(x) = \sum\limits_{n=l}^{\infty} f_n(x)$ then for all $x \in [0,1]$ we have

$$\big|S_l(x) - S(x)\big| = \sum_{n=l+1}^{\infty} f_n(x) \leq \max\{M_{l+1}, M_{l+2}, \cdots\} = \tfrac{1}{l+1}$$

since for each $x$, at most one of the terms $f_n(x)$ is non-zero.

**4:** (a) Find the Taylor series centered at 0, and its interval of convergence, for $f(x) = \dfrac{x}{x^2 - 6x + 8}$.

Solution: We have

$$f(x) = \frac{x}{x^2-6x+8} = \frac{x}{(x-2)(x-4)} = \frac{-1}{x-2} + \frac{2}{x-4} = \frac{\frac{1}{2}}{1-\frac{x}{2}} - \frac{\frac{1}{2}}{1-\frac{x}{4}}.$$

Since $\dfrac{\frac{1}{2}}{1-\frac{x}{2}} = \sum\limits_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum\limits_{n=0}^{\infty} \frac{1}{2\cdot 2^n} x^n$ when $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$, and $\dfrac{\frac{1}{2}}{1-\frac{x}{4}} = \sum\limits_{n=0}^{\infty} \frac{1}{2\cdot 4^n} x^n$ when $|x| < 4$, we have

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{2\cdot 2^n} x^n - \sum_{n=0}^{\infty} \tfrac{1}{2\cdot 4^n} x^n = \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{1}{2^n} - \tfrac{1}{4^n}\right) x^n$$

when $|x| < 2$.

(b) Find the Taylor series centered at $\frac{\pi}{4}$, and its interval of convergence, for $f(x) = \sin x \cos x$.

Solution: We provide two solutions. The first solution uses the known Taylor series for $\cos x$. We have

$$\begin{aligned} f(x) &= \sin x \cos x = \tfrac{1}{2}\sin 2x = \tfrac{1}{2}\cos\left(2x - \tfrac{\pi}{2}\right) = \tfrac{1}{2}\cos\left(2\left(x - \tfrac{\pi}{4}\right)\right) \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(2\left(x-\tfrac{\pi}{4}\right)\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!}\left(x - \tfrac{\pi}{4}\right)^{2n} \end{aligned}$$

for all $x \in \mathbb{R}$.

The second solution uses the formula for the coefficients of the Taylor series. We have $f(x) = \frac{1}{2}\sin 2x$, $f'(x) = \cos 2x$, $f''(x) = -2\sin 2x$, $f'''(x) = -4\cos 2x$, $f''''(x) = 8\sin 2x$ and so on. Put in $x = \frac{\pi}{4}$ to get $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, $f'\left(\frac{\pi}{4}\right) = 0$, $f''\left(\frac{\pi}{4}\right) = -2$, $f'''\left(\frac{\pi}{4}\right) = 0$, $f''''\left(\frac{\pi}{4}\right) = 8$ and so on. In general, the odd-order derivatives at 0 are all zero, that is $f^{(2n+1)}(0) = 0$, and the even-order derivatives are given by $f^{(2n)}(0) = (-1)^n 2^{2n-1}$. Thus the coefficients of the Taylor series are given by $c_{2n+1} = 0$ and $c_{2n} = \dfrac{(-1)^n 2^{2n-1}}{(2n)!}$, so the Taylor series is $T(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n-1}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$. To find the interval of convergence, let $a_n = \dfrac{(-1)^n 2^{2n-1}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{4\left|x - \frac{\pi}{4}\right|^2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$, so $\sum a_n$ converges for all $x \in \mathbb{R}$.

(c) Let $0 < a < b$. Note that $\mathbb{Q} \cap [a,b]$ is countable, say $\mathbb{Q} \cap [a,b] = \{q_1, q_2, q_3, \cdots\}$. Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} q_n x^n$.

Solution: Since $0 < a \leq q_n \leq b$, we have $0 < \sqrt[n]{a} \leq \sqrt[n]{q_n} \leq \sqrt[n]{b}$ for all $n$, and since $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1 = \lim\limits_{n\to\infty} \sqrt[n]{b}$ we have $\lim\limits_{n\to\infty} \sqrt[n]{q_n} = 1$ by the Squeeze Theorem. Thus the radius of convergence is $R = 1/\lim\limits_{n\to\infty} \sqrt[n]{q_n} = 1$. When $x = \pm 1$, $\lim\limits_{n\to\infty} q_n x^n$ does not exist and so $\sum q_n x^n$ diverges. Thus the interval of convergence is $I = (-1, 1)$.

**5:** (a) Find the 4$^{\text{th}}$ Taylor polynomial centered at 0 for $f(x) = \dfrac{\ln(1+x)}{e^{2x}}$.

Solution: We have

$$\begin{aligned} f(x) &= e^{-2x}\ln(1+x) \\ &= \Big(1+(-2x)+\tfrac{1}{2!}(-2x)^2+\tfrac{1}{3!}(-2x)^3+\tfrac{1}{4!}(-2x)^4+\cdots\Big)\Big(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\cdots\Big) \\ &= \Big(1-2x+2x^2-\tfrac{4}{3}x^3+\tfrac{2}{3}x^4-\cdots\Big)\Big(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\cdots\Big) \\ &= x-\big(\tfrac{1}{2}+2\big)x^2+\big(\tfrac{1}{3}+1+2\big)x^3-\big(\tfrac{1}{4}+\tfrac{2}{3}+1+\tfrac{4}{3}\big)x^4+\cdots \\ &= x-\tfrac{5}{2}x^2+\tfrac{10}{3}x^3-\tfrac{13}{4}x^4+\cdots \end{aligned}$$

so the Taylor polynomial of degree 4 is $T_4(x) = x-\frac{5}{2}x^2+\frac{10}{3}x^3-\frac{13}{4}x^4$.

(b) Find the 7$^{\text{th}}$ Taylor polynomial centered at 0 for $f(x) = \sec(\sqrt{2}\,x)$.

Solution: $f(x) = \dfrac{1}{\cos(\sqrt{2}\,x)} = \dfrac{1}{1-\frac{1}{2}(2\,x^2)+\frac{1}{24}(4x^4)-\frac{1}{720}(8x^6)+\cdots} = \dfrac{1}{1-x^2+\frac{1}{6}x^4-\frac{1}{90}x^6+\cdots}$. We perform long division:

$$\begin{array}{r|l} & 1+x^2+\tfrac{5}{6}x^4+\tfrac{61}{90}x^6+\cdots \\ \hline 1-x^2+\tfrac{1}{6}x^4-\tfrac{1}{90}x^6+\cdots & 1+0x^2+0x^4+0x^6+\cdots \\ & 1-x^2+\tfrac{1}{6}x^4-\tfrac{1}{90}x^6+\cdots \\ \hline & x^2-\tfrac{1}{6}x^4+\tfrac{1}{90}x^6+\cdots \\ & x^2-x^4+\tfrac{1}{6}x^6+\cdots \\ \hline & \tfrac{5}{6}x^4-\tfrac{14}{90}x^6+\cdots \\ & \tfrac{5}{6}x^4-\tfrac{5}{6}x^6+\cdots \\ \hline & \tfrac{61}{90}x^6+\cdots \end{array}$$

so $T_7(x) = 1+x^2+\frac{5}{6}x^4+\frac{61}{90}x^6$.

(c) Let $f(x) = x^3+x+1$. Note that $f$ is increasing with $f(0)=1$, and let $g(x) = f^{-1}(x)$. Find the 6$^{\text{th}}$ Taylor polynomial centered at 1 for the inverse function $g(x)$.

Solution: Say $g(y) = a_0+a_1(y-1)+a_2(y-1)^2+a_3(y-1)^3+\cdots$. Then

$$\begin{aligned} x = g\big(f(x)\big) = g(x^3+x+1) &= a_0+a_1(x+x^3)+a_2(x+x^3)^2+a_3(x+x^3)^3+\cdots \\ &= a_0+a_1(x+x^3)+a_2(x^2+2x^4+x^6)+a_3\big(x^3+3x^5+\cdots\big) \\ &\quad +a_4\big(x^4+4x^6+\cdots\big)+a_5\big(x^5+\cdots\big)+a_6\big(x^6+\cdots\big)+\cdots \\ &= a_0+a_1x+a_2x^2+(a_3+a_1)x^3+(a_4+2a_2)x^4+(a_5+3a_3)x^5+(a_6+4a_4+a_2)x^6+\cdots \end{aligned}$$

Comparing coefficients, we see that $a_0=0$, $a_1=1$, $a_2=0$, $a_3=-a_1=-1$, $a_4=-2a_2=0$, $a_5=-3a_3=3$ and $a_6=-4a_4-a_2=0$, and so the 6$^{\text{th}}$ Taylor polynomial is $T_6(x) = (x-1)-(x-1)^3+3(x-1)^5$.

**6:** (a) Let $f(x) = (8 + x^3)^{2/3}$. Find $f^{(9)}(0)$, the $9^{\text{th}}$ derivative of $f$ at 0.

Solution: $f(x) = (8 + x^3)^{2/3} = 4\big(1 + \frac{x^3}{8}\big)^{2/3} = 4\Big(1 + \frac{2}{3}\frac{x^3}{8} + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\left(\frac{x^3}{8}\right)^2 + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\left(\frac{x^3}{8}\right)^3 + \cdots\Big)$, so $c_9 = \frac{4\cdot 2\cdot 1\cdot 4}{3^3\cdot 3!\cdot 8^3} = \frac{1}{3^4\,2^5}$ and $f^{(9)}(0) = 9!\,c_9 = \frac{9!}{3^4\,2^5} = 140$.

(b) Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{x\,e^{x^2} - \sin x}{x - \tan^{-1} x}$.

Solution: $\displaystyle\lim_{x\to 0} \frac{x\,e^{x^2} - \sin x}{x - \tan^{-1} x} = \lim_{x\to 0} \frac{x\,(1 + x^2 + \frac{1}{2}x^4 + \cdots) - (x - \frac{1}{6}x^3 + \cdots)}{x - (x - \frac{1}{3}x^3 + \cdots)} \lim_{x\to 0} \frac{\frac{7}{6}x^3 + \cdots}{\frac{1}{3}x^3 + \cdots} = \frac{7}{2}$.

(c) Suppose that there exists a function $y = f(x)$, whose Taylor series centered at 0 has a positive radius of convergence, such that $\frac{1}{2}\,y'' + y' - 3y = x + 1$ with $y(0) = 1$ and $y'(0) = 2$. Find the Taylor polynomial of degree 5 centred at 0 for $f(x)$.

Solution: Let $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$. Then $y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$ and $y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots$. So we have

$$\begin{aligned}
0 &= \tfrac{1}{2}\,y'' + y' - 3y - x - 1 \\
&= (c_2 + 3c_3x + 6c_4x^2 + 10c_5x^3 + \cdots) + (c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots) \\
&\quad - (3c_0 + 3c_1x + 3c_2x^2 + 3c_3x^3 + \cdots) - x - 1 \\
&= (c_2 + c_1 - 3c_0 - 1) + (3c_3 + 2c_2 - 3c_1 - 1)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots
\end{aligned}$$

Since $y(0) = 1$ and $y'(0) = 2$ we have $c_0 = 1$ and $c_2 = 2$. Put these values in the above equation to get

$$0 = (c_2 - 2) + (3c_3 + 2c_2 - 7)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots$$

For $y$ to be a solution, all the coefficients must be zero, so we have

$$\begin{aligned}
&(c_2 - 2) = 0 \Longrightarrow c_2 = 2 \\
&(3c_3 + 2c_2 - 7) = 0 \Longrightarrow 3c_3 = 7 - 2c_2 = 3 \Longrightarrow c_3 = 1 \\
&(6c_4 + 3c_3 - 3c_2) = 0 \Longrightarrow 6c_4 = 3c_2 - 3c_3 = 3 \Longrightarrow c_4 = \tfrac{1}{2} \\
&(10c_5 + 4c_4 - 3c_3) = 0 \Longrightarrow 10c_5 = 3c_3 - 4c_4 = 1 \Longrightarrow c_5 = \tfrac{1}{10}\,.
\end{aligned}$$

Thus the Taylor polynomial of degree 5 centered at 0 is

$$T_5(x) = 1 + 2x + 2x^2 + x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{10}x^5\,.$$

**7:** Estimate each of the following numbers so that the error is at most $\frac{1}{1000}$.

(a) $\sqrt[5]{e}$

Solution: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$, so we have

$$\sqrt[5]{e} = e^{\frac{1}{5}} = 1 + \frac{1}{5} + \frac{1}{5^2 2!} + \frac{1}{5^3 3!} + \frac{1}{5^4 4!} + \cdots \cong 1 + \frac{1}{5} + \frac{1}{5^2 2!} + \frac{1}{5^3 3!} = 1 + \frac{1}{5} + \frac{1}{50} + \frac{1}{750} = \frac{916}{750}$$

with error

$$\begin{aligned} E &= \tfrac{1}{5^4 4!} + \tfrac{1}{5^5 5!} + \tfrac{1}{5^6 6!} + \cdots = \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5\cdot 5} + \tfrac{1}{5^2\cdot 5\cdot 6} + \tfrac{1}{5^3\cdot 5\cdot 6\cdot 7} + \cdots\right) \\ &\le \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5^2} + \tfrac{1}{5^4} + \tfrac{1}{5^6} + \cdots\right) = \tfrac{1}{5^4 4!}\,\tfrac{1}{1-\frac{1}{25}} = \tfrac{1}{5^4 4!}\,\tfrac{25}{24} = \tfrac{1}{13200} \end{aligned}$$

where we used the C.T. and the formula for the sum of a geometric series.

(b) $\ln(4/5)$

Solution: We provide two solutions. For the first solution, we use $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$. We have

$$\ln\left(\tfrac{4}{5}\right) = \ln\left(1 - \tfrac{1}{5}\right) = -\tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} - \tfrac{1}{3\cdot 5^3} - \tfrac{1}{4\cdot 5^4} - \cdots \cong -\tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} - \tfrac{1}{3\cdot 5^3} = -\tfrac{1}{5} - \tfrac{1}{50} - \tfrac{1}{375} = -\tfrac{167}{750}$$

with error

$$E = \tfrac{1}{4\cdot 5^4} + \tfrac{1}{5\cdot 5^5} + \tfrac{1}{6\cdot 5^6} + \cdots < \tfrac{1}{4\cdot 5^4} + \tfrac{1}{4\cdot 5^5} + \tfrac{1}{4\cdot 5^6} + \cdots = \frac{\frac{1}{4\cdot 5^4}}{1 - \frac{1}{5}} = \tfrac{1}{4\cdot 5^4}\cdot\tfrac{5}{4} = \tfrac{1}{4^2\cdot 5^3} = \tfrac{1}{2000}\,,$$

where we used the C.T. and the formula for the sum of a geometric series.

For the second solution, we use $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$. We have

$$\ln\tfrac{4}{5} = -\ln\tfrac{5}{4} = -\ln\left(1 + \tfrac{1}{4}\right) = \left(-\tfrac{1}{4} + \tfrac{1}{2\cdot 4^2} - \tfrac{1}{3\cdot 4^3} + \tfrac{1}{4\cdot 4^4} - \cdots\right) \cong -\tfrac{1}{4} + \tfrac{1}{32} - \tfrac{1}{192} = -\tfrac{43}{192}$$

with error $E \le \frac{1}{4\cdot 4^4} < \frac{1}{1000}$ by the A.S.T.

(c) $\displaystyle\int_0^1 \sqrt{4+x^3}\,dx$

Solution: Using the Binomial Series, we have

$$\begin{aligned} &\sqrt{4+x^3}\,dx = 2\left(1 + \frac{x^3}{4}\right)^{1/2} \\ &= 2\left(1 + \tfrac{1}{2}\left(\frac{x^3}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x^3}{4}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x^3}{4}\right)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}\left(\frac{x^3}{4}\right)^4 + \cdots\right) \\ &= 2 + \tfrac{1}{4}\,x^3 - \tfrac{1}{2\cdot 2!\cdot 4^2}\,x^6 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3}\,x^9 - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4}\,x^{12} + \cdots \end{aligned}$$

and so

$$\begin{aligned} \int_0^1 \sqrt{4+x^3}\,dx &= \left[2\,x + \tfrac{1}{4\cdot 4}\,x^4 - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7}\,x^7 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10}\,x^{10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13}\,x^{13} + \cdots\right]_0^1 \\ &= 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13} + \cdots \\ &\cong 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} = 2 + \tfrac{1}{16} - \tfrac{1}{448} = \tfrac{923}{448} \end{aligned}$$

with absolute error $E \le \frac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} = \frac{1}{5120}$ by the A.S.T.

To be rigorous, we should justify our application of the A.S.T. When $a_n = \dfrac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)}{2^{n-1}\cdot n!\cdot 4^n\cdot(3n+1)}$ we have $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2n-1}{2\cdot(n+1)\cdot 4\cdot(3n+4)} < \dfrac{2n+2}{2\cdot(n+1)\cdot 4\cdot(3n+4)} = \dfrac{1}{4\cdot(3n+4)}$. Since $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ we know that $\{|a_n|\}$ is decreasing, and since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ we know that $\sum a_n$ converges by the R.T. so $\displaystyle\lim_{n\to\infty}|a_n| = 0$ by the D.T. Thus we can indeed apply the A.S.T.

**8:** Find the exact value of each of the following sums.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}$

Solution: For all $x$ we have $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $x\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$. Differentiate to get $(x+1)\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)x^n}{n!}$, so $(x^2+x)\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)x^{n+1}}{n!}$. Differentiate again to get $(x^2+3x+1)\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)^2x^n}{n!}$. Put in $x=1$ to get $5\,e=\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)^2}{n!}=1+\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}$. Thus $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}=5\,e-1$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(2n+1)2^n}$

Solution: Let $S=\displaystyle\sum_{n=1}^{\infty}\frac{n}{(2n+1)2^n}=\frac{1}{3\cdot 2^1}+\frac{2}{5\cdot 2^2}+\frac{3}{7\cdot 2^3}+\cdots$. For $|x|<1$ we have $\dfrac{1}{1-x^2}=1+x^2+x^4+x^6+\cdots$. Integrate both sides to get $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$. Divide both sides by $x$ and the differentiate to get $\dfrac{\frac{x}{1-x^2}-\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}{x^2}=\frac{2x}{3}+\frac{4x^3}{5}+\frac{6x^5}{7}+\cdots$. Multiply by $x$ to get $\frac{1}{1-x^2}-\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)=\frac{2x^3}{3}+\frac{4x^4}{5}+\frac{6x^6}{7}+\cdots$. Put in $x=\frac{1}{\sqrt{2}}$ to get $2-\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=\frac{2}{3\cdot 2}+\frac{4}{5\cdot 2^2}+\frac{6}{7\cdot 2^3}+\cdots=2S$. Thus

$$S=1-\tfrac{1}{2\sqrt{2}}\ln\left(\tfrac{\sqrt{2}+1}{\sqrt{2}-1}\right)=1-\tfrac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right).$$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}$

Solution: For $|x|<1$ we have $\displaystyle\sum_{n=0}^{\infty}(-1)^nx^{3n}=\frac{1}{1+x^3}$, so $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^nx^{3n+1}}{3n+1}=\int_0^x\frac{dt}{1+t^3}$. By Abel's Theorem we can put in $x=1$ to get $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}=\int_0^1\frac{dt}{1+t^3}$. Thus

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}=\tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n-2}=\tfrac{1}{2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}=\tfrac{1}{2}+\int_0^1\frac{dt}{t^3+1}.$$

To get $\dfrac{1}{t^3+1}=\dfrac{A}{t+1}+\dfrac{B(2t-1)+C}{t^2-t+1}$, we need $A(t^2-t+1)+B(2t^2+t-1)+C(t+1)=1$. Equate coefficients to get the three equations $A+2B=0$, $-A+B+C=0$ and $A-B+C=1$. Solve these to get $A=\frac{1}{3}$, $B=-\frac{1}{6}$ and $C=\frac{1}{2}$. Thus we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3n-2}&=\tfrac{1}{2}+\int_0^1\frac{dt}{t^3+1}=\tfrac{1}{2}+\int_0^1\frac{\frac{1}{3}}{t+1}-\frac{\frac{1}{6}(2t-1)+\frac{1}{2}}{t^2-t+1}\,dt\\&=\tfrac{1}{2}+\left[\tfrac{1}{3}\ln(t+1)-\tfrac{1}{6}\ln(t^2-t+1)+\tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{(t-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\right]_0^1\\&=\tfrac{1}{2}+\tfrac{1}{3}\ln 2+\tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{1}{\sqrt{3}}-\tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{1}{\sqrt{3}}=\tfrac{1}{2}+\tfrac{1}{3}\ln 2+\tfrac{\pi}{3\sqrt{3}}.\end{aligned}$$

**9:** (Dirichlet's Tests for Convergence)

(a) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences in $\mathbb{R}$. Suppose there exists $M \geq 0$ such that $\left|\sum\limits_{n=1}^{\ell} a_n\right| \leq M$ for all $\ell \in \mathbb{Z}^+$ and suppose that $(b_n)_{n\geq 1}$ is decreasing with $b_n \to 0$. Show that $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

Solution: I may include a solution later.

(b) Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n} \sin nx$ converges for all $x \in \mathbb{R}$.

Solution: I may include a solution later.

(c) Let $\emptyset \neq A \subseteq \mathbb{R}$, and let $f_n, g_n : A \to \mathbb{R}$ for all $n \in \mathbb{Z}^+$. Suppose that there exists $M \geq 0$ such that $\left|\sum\limits_{n=1}^{\ell} f_n(x)\right| \leq M$ for all $\ell \in \mathbb{Z}^+$ and all $x \in A$, and suppose that $\big(g_n(x)\big)_{n\geq 1}$ is decreasing for all $x \in A$ with $g_n \to 0$ uniformly in $A$. Prove that $\sum\limits_{n=1}^{\infty} f_n g_n$ converges uniformly on $A$.

Solution: I may include a solution later.

(d) Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n} \cos nx$ converges uniformly on every closed interval $[a,b] \subseteq (0, 2\pi)$.

Solution: I may include a solution later.