MATH 148 Calculus 2, Exercises for Chapter 7

1: (a) Define $f_n : [0,\infty) \to \mathbb{R}$ by $f_n(x) = nxe^{-nx}$. Find the pointwise limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0,\infty)$.

(b) Define $f_n : [0,\infty) \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+nx^2}$. Find the pointwise limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0,\infty)$.

(c) Define $f_n : [0,\infty] \to \mathbb{R}$ by $f_n(x) = \frac{x+n}{x+4n}$. Show that (f_n) converges uniformly on $[0,r]$ for every $r > 0$ but that (f_n) does not converge uniformly on $[0,\infty)$.

2: (a) Find $\displaystyle\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx$ and $\displaystyle\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx$.

(b) Find $\displaystyle\int_1^4 \lim_{n\to\infty} \frac{\tan^{-1}(nx)}{x}\,dx$ and $\displaystyle\lim_{n\to\infty}\int_1^4 \frac{\tan^{-1}(nx)}{x}\,dx$.

(c) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}$ converges uniformly on $\mathbb{R}$ and find $\displaystyle\int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}\,dx$.

(d) Show that $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly on any closed interval $[a,b]$.

3: Determine which of the following statements are true for all sequences of functions (f_n) and (g_n) and all $E \subseteq \mathbb{R}$.

(a) If (f_n) and (g_n) converge uniformly on E then $(f_n g_n)$ converge uniformly on E.

(b) Show that if (f_n) and (g_n) converge uniformly on E and f and g are bounded on E then $(f_n g_n)$ converges uniformly on E.

(c) If (f_n) converges uniformly on (a,b) and pointwise on $[a,b]$ then (f_n) converges uniformly on $[a,b]$.

(d) If each f_n is continuous on $[a,b]$ and $\sum f_n$ converges uniformly on $[a,b]$ then $\sum M_n$ converges, where $M_n = \max\big\{|f_n(x)|\big|a \le x \le b\big\}$.

4: (a) Find the Taylor series centred at 0, and its interval of convergence, for $f(x) = \dfrac{x}{x^2-6x+8}$.

(b) Find the Taylor series centred at $\frac{\pi}{4}$, and its interval of convergence, for $f(x) = \sin x \cos x$.

(c) Let $0 < a < b$. Note that $\mathbb{Q} \cap [a,b]$ is countable, say $\mathbb{Q} \cap [a,b] = \{q_1, q_2, q_3, \cdots\}$. Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} q_n x^n$.

5: (a) Find the 4$^{\text{th}}$ Taylor polynomial centred at 0 for $f(x) = \dfrac{\ln(1+x)}{e^{2x}}$.

(b) Find the 7$^{\text{th}}$ Taylor polynomial centred at 0 for $f(x) = \sec(\sqrt{2}\,x)$.

(c) Let $f(x) = x^3 + x + 1$. Note that f is increasing with $f(0) = 1$, and let $g(x) = f^{-1}(x)$. Find the 6$^{\text{th}}$ Taylor polynomial centred at 1 for the inverse function $g(x)$.

6: (a) Let $f(x) = (8+x^3)^{2/3}$. Find $f^{(9)}(0)$, the 9$^{\text{th}}$ derivative of f at 0.

(b) Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{x\,e^{x^2} - \sin x}{x - \tan^{-1} x}$.

(c) Suppose that there exists a function $y = f(x)$, whose Taylor series centred at 0 has a positive radius of convergence, such that $\frac{1}{2}\,y'' + y' - 3y = x + 1$ with $y(0) = 1$ and $y'(0) = 2$. Find the Taylor polynomial of degree 5 centred at 0 for $f(x)$.

7: Estimate each of the following numbers so that the error is at most $\frac{1}{1000}$.

(a) $\sqrt[5]{e}$ (b) $\ln(4/5)$ (c) $\displaystyle\int_0^1 \sqrt{4+x^3}\,dx$

8: Find the exact value of each of the following sums.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n+1)2^n}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n-2}$

9: (Dirichlet's Tests for Convergence)

(a) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences in $\mathbb{R}$. Suppose there exists $M \geq 0$ such that $\left|\sum\limits_{n=1}^{\ell} a_n\right| \leq M$ for all $\ell \in \mathbb{Z}^+$ and suppose that $(b_n)_{n\geq 1}$ is decreasing with $b_n \to 0$. Show that $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

(b) Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n} \sin nx$ converges for all $x \in \mathbb{R}$.

(c) Let $\emptyset \neq A \subseteq \mathbb{R}$, and let $f_n, g_n : A \to \mathbb{R}$ for all $n \in \mathbb{Z}^+$. Suppose that there exists $M \geq 0$ such that $\left|\sum\limits_{n=1}^{\ell} f_n(x)\right| \leq M$ for all $\ell \in \mathbb{Z}^+$ and all $x \in A$, and suppose that $\big(g_n(x)\big)_{n\geq 1}$ is decreasing for all $x \in A$ with $g_n \to 0$ uniformly in A. Prove that $\sum\limits_{n=1}^{\infty} f_n g_n$ converges uniformly on A.

(d) Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n} \cos nx$ converges uniformly on every closed interval $[a, b] \subseteq (0, 2\pi)$.