MATH 148 Calculus 2, Solutions to the Exercises for Chapter 6

1: (a) Let $a_1 = 6$ and for $n \geq 1$ let $a_{n+1} = 1 + 2^{a_n/3}$. Determine whether $\{a_n\}$ converges, and if so find the limit.

Solution: Suppose for now that $\{a_n\}$ does converge and say $\lim\limits_{n\to\infty} a_n = l$. Then by taking the limit on both sides of the recursion formula $a^{n+1} = 1 + 2^{a_n/3}$, we obtain $l = 1 + 2^{l/3}$. By inspection $l = 3$ and $l = 9$ are two solutions (to find these two solutions, sketch the graphs of $y = x$ and $y = 1 + 2^{x/3}$ on the same grid). We can prove that these are the only two solutions as follows. Let $f(x) = 1 + 2^{x/3} - x$. Then $f'(x) = \frac{\ln 2}{3} \cdot 2^{x/3} - 1$, so $f'(x) = 0$ when $\frac{\ln 2}{3} \cdot 2^{x/3} = 1$, that is when $x = a$ where $a = 3\log_2(3/\ln 2)$, and note that we have $f'(x) < 0$ when $x < a$ and $f'(x) > 0$ when $x > a$. Thus f is decreasing on $(-\infty, a]$ and increasing on $[a, \infty)$, and so f has at most two roots. Thus if $\{a_n\}$ does converge, the limit must be $l = 3$ or $l = 9$.

We claim that $9 > a_n > a_{n+1} > 3$ for all $n \geq 1$. We have $a_1 = 6$ and $a_2 = 5$, so the claim is true when $n = 1$. Suppose it is true when $n = k$. Then we have $9 > a_k > a_{k+1} > 3 \Longrightarrow 3 > a_n/3 > a_{n+1}/3 > 1 \Longrightarrow 8 > 2^{a_n/3} > 2^{a+n+1/3} > 2 \Longrightarrow 9 > 1 + 2^{a_n/3} > 1 + 2^{a_{n+1}/3} > 3$, that is $9 > a_{k+1} > a_{k+2} > 3$, and so the claim is true when $n = k + 1$. By induction, the claim is true for all $n \geq 1$. Thus $\{a_n\}$ is decreasing and is bounded below by 3, so it converges by the MCT. As shown above, the limit must be 3 or 9. Since $a_1 = 6$ and $\{a_n\}$ is decreasing, its limit must be 3.

(b) Let $a_1 = \frac{7}{2}$ and for $n \geq 1$ let $a_{n+1} = \dfrac{6}{5 - a_n}$. Determine whether $\{a_n\}$ converges, and if so find the limit.

Solution: Suppose for now that $\{a_n\}$ does converge, and say $\lim\limits_{n\to\infty} a_n = l$. Taking the limit on both sides of the recursion formula $a_{n+1} = \dfrac{6}{5 - a_n}$ gives $l = \frac{6}{5-l} \Longrightarrow 5l - l^2 = 6 \Longrightarrow l^2 - 5l + 6 = 0 \Longrightarrow (l-2)(l-3) = 0$, and so we must have $l = 2$ or $l = 3$.

We claim that $a_n < a_{n+1} < 2$ for all $n \geq 4$. We have $a_1 = \frac{7}{2}$, $a_2 = 4$, $a_3 = 6$, $a_4 = -6$ and $a_5 = \frac{6}{11}$, so the claim is true when $n = 4$. Suppose the claim is true when $n = k$. Then we have $a_k < a_{k+1} < 2 \Longrightarrow -a_k > -a_{k+1} > -2 \Longrightarrow 5 - a_k > 5 - a_{k+1} > 3 \Longrightarrow \frac{1}{5-a_k} < \frac{1}{5-a_{k+1}} < \frac{1}{3} \Longrightarrow \frac{6}{5-a_k} < \frac{6}{5-a_{k+1}} < 2$, that is $a_{k+1} < a_{k+2} < 2$, so the claim is true when $n = k+1$. By induction, the claim is true for all $n \geq 4$. Thus $\{a_n\}_{n\geq 4}$ is increasing and is bounded above by 2, so $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n \leq 2$ by the MCT. We showed above that the limit must be 2 or 3, and so we must have $\lim\limits_{n\to\infty} a_n = 2$.

(c) Let $(x_k)_{k\geq 0}$ be a sequence in $\mathbb{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k^2}$ for all $k \geq 1$. Show that (x_k) converges in $\mathbb{R}$.

Solution: Notice that for all $k \geq 2$ we have $\frac{1}{k^2} \leq \frac{1}{(k-1)\,k} = \frac{1}{k-1} - \frac{1}{k}$. It follows that for $1 \leq k < l$ we have

$$\begin{aligned}
|x_k - x_l| &= \left|x_k - x_{k+1} + x_{k+1} - x_{k+2} + x_{k+2} - x_{k+3} + \cdots - x_{l-1} + x_{l-1} - x_l\right| \\
&\leq |x_k - x_{k+1}| + |x_{k+1} - x_{k+2}| + |x_{k+2} - x_{k+3}| + \cdots + |x_{l-1} - x_l| \\
&\leq \tfrac{1}{(k+1)^2} + \tfrac{1}{(k+2)^2} + \tfrac{1}{(k+3)^2} + \cdots + \tfrac{1}{(l-1)^2} + \tfrac{1}{l^2} \\
&\leq \tfrac{1}{k(k+1)} + \tfrac{1}{(k+1)(k+2)} + \tfrac{1}{(k+2)(k+3)} + \cdots + \tfrac{1}{(l-2)(l-1)} + \tfrac{1}{(l-1)\,l} \\
&= \tfrac{1}{k} - \tfrac{1}{k+1} + \tfrac{1}{k+1} - \tfrac{1}{k+2} + \tfrac{1}{k+2} - \tfrac{1}{k+3} + \cdots - \tfrac{1}{l-1} + \tfrac{1}{l-1} - \tfrac{1}{l} \\
&= \tfrac{1}{k} - \tfrac{1}{l} \leq \tfrac{1}{k}.
\end{aligned}$$

Let $\epsilon > 0$. Choose $m \in \mathbb{Z}$ with $m \geq \frac{1}{\epsilon}$. For $k, l \geq m$ say with $k \leq l$, if $k = l$ then $|x_k - x_l| = 0$ and if $k < l$ then, as shown above, $|x_k - x_l| \leq \frac{1}{k} \leq \frac{1}{m} \leq \epsilon$. Thus (x_k) is a Cauchy sequence, and so it converges by the Cauchy Criterion.

2: Determine which of the following series converge.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{2n^2+1}$

Solution: For $n \geq 1$ we have $0 \leq \dfrac{\sqrt{n}}{2n^2+1} \leq \dfrac{\sqrt{n}}{2n^2} = \dfrac{1}{2n^{3/2}}$, and we know that $\sum \dfrac{1}{2n^{3/2}}$ converges (since it is a constant multiple of the p-series with $p = \frac{3}{2}$), and so $\sum \dfrac{\sqrt{n}}{2n^2+1}$ converges by the C.T.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \, 2^{1/n}$

Solution: Let $a_n = (-1)^n 2^{1/n}$. Then $|a_n| = 2^{1/n} \longrightarrow 2^0 = 1$. Since $|a_n| \not\to 0$, we know that $a_n \not\to 0$, and so $\sum a_n$ diverges by the D.T.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n^n}{(2n)!}$

Solution: Let $a_n = \dfrac{n!n^n}{(2n)!}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!(n+1)^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{n!n^n} = \dfrac{(n+1)^2}{(2n+2)(2n+1)} \cdot \left(\dfrac{n+1}{n}\right)^n \to \dfrac{e}{4} < 1$, so $\sum a_n$ converges by the R.T.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(n \sin^{-1}\left(\tfrac{1}{n}\right) - 1\right)$.

Solution: Let $a_n = \left(n \sin^{-1}\left(\frac{1}{n}\right) - 1\right)$ and let $b_n = \frac{1}{n^2}$. Then using l'Hôpital's Rule twice, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n \sin^{-1}\left(\frac{1}{n}\right) - 1}{\frac{1}{n^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}\sin^{-1} x - 1}{x^2} = \lim_{x\to 0^+} \frac{\sin^{-1} x - x}{x^3} \\
&= \lim_{x\to 0^+} \frac{(1-x^2)^{-1/2} - 1}{3x^2} = \lim_{x\to 0^+} \frac{x(1-x^2)^{-3/2}}{6x} = \lim_{x\to 0^+} \frac{(1-x^2)^{-3/2}}{6} = \tfrac{1}{6}\,.
\end{aligned}$$

Since $\sum b_n$ converges (its a p-series with $p = 2$), $\sum a_n$ converges too by the L.C.T.

3: Find the sum of each of the following series, if the sum exists.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1}+3^n}{6^{n-1}}$

Solution: $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1}+3^n}{6^{n-1}} = \sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{6^{n-1}} + \sum_{n=0}^{\infty} \frac{3^n}{6^{n-1}} = \frac{-12}{1+\frac{1}{3}} + \frac{6}{1-\frac{1}{2}} - 9 + 12 = 3.$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n^2-4}$

Solution: The l^{th} partial sum is

$$\begin{aligned} S_l &= \sum_{n=3}^{l} \frac{2}{n^2-4} = \sum_{n=3}^{l} \left(\frac{\frac{1}{2}}{n-2} - \frac{\frac{1}{2}}{n+2} \right) \\ &= \tfrac{1}{2}\Big(\left(\tfrac{1}{1}-\tfrac{1}{5}\right) + \left(\tfrac{1}{2}-\tfrac{1}{6}\right) + \left(\tfrac{1}{3}-\tfrac{1}{7}\right) + \left(\tfrac{1}{4}-\tfrac{1}{8}\right) + \cdots \\ &\quad + \left(\tfrac{1}{l-5}-\tfrac{1}{l-1}\right) + \left(\tfrac{1}{l-4}-\tfrac{1}{l}\right) + \left(\tfrac{1}{l-3}-\tfrac{1}{l+1}\right) + \left(\tfrac{1}{l-2}-\tfrac{1}{l+2}\right)\Big) \\ &= \tfrac{1}{2}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{l-1}-\tfrac{1}{l}-\tfrac{1}{l+1}-\tfrac{1}{l+2}\right), \end{aligned}$$

so the sum is $S = \lim\limits_{l\to\infty} S_l = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{25}{24}$.

(c) $\displaystyle\sum_{n=-1}^{\infty} e^{-(n\ln 2)/2}$

Solution: Note that $e^{-(n\ln 2)/2} = \left(\frac{1}{\sqrt{2}}\right)^n n$, so this is a geometric series, and we have

$$\sum_{n=-1}^{\infty} e^{-(n\ln 2)/2} = \sum_{n=-1}^{\infty} \left(\tfrac{1}{\sqrt{2}}\right)^n = \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} = \frac{2}{\sqrt{2}-1} = 2(\sqrt{2}+1)\,.$$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{6n^2}{n^6-1}$

Solution: Note that $\displaystyle\frac{6n^2}{n^6-1} = \frac{3n^2}{n^3-1} - \frac{3n^2}{n^3+1} = \frac{1}{n-1} + \frac{2n+1}{n^2+n+1} - \frac{1}{n+1} - \frac{2n-1}{n^2-n+1}$, so the l^{th} partial sum is

$$\begin{aligned} S_l &= \sum_{n=2}^{l} \left(\left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{2\left(n-\frac{1}{2}\right)}{\left(n-\frac{1}{2}\right)^2+\frac{3}{4}} \right) \right) \\ &= \left(\left(\tfrac{1}{1}-\tfrac{1}{3}\right) + \left(\tfrac{1}{2}-\tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{l-2}-\tfrac{1}{l}\right) + \left(\tfrac{1}{l-1}-\tfrac{1}{l+1}\right) \right) \\ &\quad + 2\left(\left(\frac{\frac{5}{2}}{\left(\frac{5}{2}\right)^2+\frac{3}{4}} - \frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+\frac{3}{4}} \right) + \left(\frac{\frac{7}{2}}{\left(\frac{7}{2}\right)^2+\frac{3}{4}} - \frac{\frac{5}{2}}{\left(\frac{5}{2}\right)^2+\frac{3}{4}} \right) + \cdots \right. \\ &\quad \left. + \left(\frac{l-\frac{1}{2}}{\left(l-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{l-\frac{3}{2}}{\left(l-\frac{3}{2}\right)^2+\frac{3}{4}} \right) + \left(\frac{l+\frac{1}{2}}{\left(l+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{l-\frac{1}{2}}{\left(l-\frac{1}{2}\right)^2+\frac{3}{4}} \right) \right) \\ &= \left(1+\tfrac{1}{2}-\tfrac{1}{l}-\tfrac{1}{l+1}\right) + 2\left(-\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+\frac{3}{4}} + \frac{l+\frac{1}{2}}{\left(l+\frac{1}{2}\right)^2+\frac{3}{4}} \right) \end{aligned}$$

so the sum is $S = \lim\limits_{l\to\infty} S_l = 1+\frac{1}{2} - 2\dfrac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+\frac{3}{4}} = 1+\frac{1}{2}-1 = \frac{1}{2}$.

4: Find the sum of each of the following series, if the sum exists.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n+1)!}$

Solution: Note that $\frac{1}{n!}-\frac{1}{(n+1)!}=\frac{(n+1)-1}{(n+1)!}=\frac{n}{(n+1)!}$ and so

$$\begin{aligned}S_l&=\sum_{n=0}^{l}\tfrac{n}{(n+1)!}=\sum_{n=0}^{l}\left(\tfrac{1}{n!}-\tfrac{1}{(n+1)!}\right)\\&=\left(\tfrac{1}{0!}-\tfrac{1}{1!}\right)+\left(\tfrac{1}{1!}-\tfrac{1}{2!}\right)+\left(\tfrac{1}{2!}-\tfrac{1}{3!}\right)+\cdots+\left(\tfrac{1}{(l-1)!}-\tfrac{1}{l!}\right)+\left(\tfrac{1}{l!}-\tfrac{1}{(l+1)!}\right)\\&=\tfrac{1}{0!}-\tfrac{1}{(l+1)!}=1-\tfrac{1}{(l+1)!}\end{aligned}$$

and so $\sum_{n=0}^{\infty}\frac{n}{(n+1)!}=\lim_{l\to\infty}S_l=\lim_{l\to\infty}\left(1-\frac{1}{(l+1)!}\right)=1.$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

Solution: More generally, let us find $\displaystyle\sum_{n=1}^{\infty}n^2r^n$ where $|r|<1$. Let $S_l=\displaystyle\sum_{n=1}^{l}n^2r^n=r+2^2r^2+3^2r^3+\cdots l^2r^l$. Then $rS_l=r^2+2^2r^3+\cdots+(l-1)^2r^l+l^2r^{l+1}$ and so

$$\begin{aligned}(1-r)S_l&=r+(2^2-1^2)r^2+(3^2-2^2)r^3+\cdots+(l^2-(l-1)^2)r^l-l^2r^{l+1}\\&=\left(r+3r^2+5r^3+\cdots+(2l-1)r^l\right)-l^2r^{l+1}=T_l-l^2r^{l+1}\,,\end{aligned}$$

where $T_l=\displaystyle\sum_{n=1}^{l}(2n-1)r^n=r+3r^2+5r^3+\cdots+(2l-1)r^l$. We have $rT_l=r^2+3r^3+\cdots+(2l-3)r^l+(2l-1)r^{l+1}$ and so

$$(1-r)T_l=r+2r^2+2r^3+\cdots+2r^l-(2l-1)r^{l+1}=-r+2U_l-(2l-1)r^{l+1}\,,$$

where $U_l=\displaystyle\sum_{n=1}^{l}r^l=r+r^2+r^3+\cdots+r^l$. Since $\displaystyle\lim_{l\to\infty}U_l=\sum_{n=1}^{\infty}r^n=\frac{r}{1-r}$ we have

$$\sum_{n=1}^{\infty}(2n-1)r^n=\lim_{l\to\infty}T_l=\lim_{l\to\infty}\frac{-r+2U_l-(2l-1)r^{l+1}}{1-r}=\frac{-r+\frac{2r}{1-r}-0}{1-r}=\frac{r(1+r)}{(1-r)^2}$$

since $\displaystyle\lim_{l\to\infty}(2l-1)r^{l+1}=0$, as you can verify using l'Hôpital's Rule, and

$$\sum_{n=1}^{\infty}n^2r^n=\lim_{l\to\infty}S_l=\lim_{l\to\infty}\frac{T_l-l^2r^{l+1}}{1-r}=\frac{\frac{r(1+r)}{(1-r)^2}-0}{1-r}=\frac{r(1+r)}{(1-r)^3}$$

since $\displaystyle\lim_{l\to\infty}l^2r^{l+1}=0$. In particular, taking $r=\frac{1}{2}$ gives $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\frac{\frac{1}{2}\cdot\frac{3}{2}}{\left(\frac{1}{2}\right)^3}=6\,.$

(c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{a_{n-1}a_{n+1}}$, where $\{a_n\}$ is the Fibonacci sequence.

Solution: Note that $\frac{1}{a_{n-1}a_n}-\frac{1}{a_na_{n+1}}=\frac{a_{n+1}-a_{n-1}}{a_{n-1}a_na_{n+1}}=\frac{a_n}{a_{n-1}a_na_{n+1}}=\frac{1}{a_{n-1}a_{n+1}}$ and so

$$\begin{aligned}S_l&=\sum_{n=2}^{l}\tfrac{1}{a_{n-1}a_{n+1}}=\sum_{n=2}^{l}\left(\tfrac{1}{a_{n-1}a_n}-\tfrac{1}{a_na_{n+1}}\right)\\&=\left(\tfrac{1}{a_1a_2}-\tfrac{1}{a_2a_3}\right)+\left(\tfrac{1}{a_2a_3}-\tfrac{1}{a_3a_4}\right)+\cdots+\left(\tfrac{1}{a_{l-2}a_{l-1}}-\tfrac{1}{a_{l-1}a_l}\right)+\left(\tfrac{1}{a_{l-1}a_l}-\tfrac{1}{a_la_{l+1}}\right)\\&=\frac{1}{a_1a_2}-\frac{1}{a_la_{l+1}}=1-\frac{1}{a_la_{l+1}}\,.\end{aligned}$$

Thus $\sum_{n=2}^{\infty}\frac{1}{a_{n-1}a_{n+1}}=\lim_{l\to\infty}S_l=\lim_{l\to\infty}\left(1-\frac{1}{a_la_{l+1}}\right)=1.$

5: Given a sequence $(a_n)_{n\geq k}$, we define the infinite product $\prod_{n=k}^{\infty} a_n$ to be $\lim_{\ell\to\infty} P_\ell$ where $P_\ell = \prod_{n=k}^{\ell} a_n$, if the limit exists. Evaluate each of the following infinite products.

(a) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$

Solution: We have

$$P_l = \prod_{n=2}^{l}\left(1-\tfrac{1}{n^2}\right) = \prod_{n=2}^{l}\left(\tfrac{n^2-1}{n^2}\right) = \prod_{n=2}^{l}\tfrac{(n-1)(n+1)}{n^2} = \tfrac{1\cdot 3}{2^2}\cdot\tfrac{2\cdot 4}{3^2}\cdot\tfrac{3\cdot 5}{4^2}\cdot\tfrac{4\cdot 6}{5^2}\cdots\tfrac{(n-2)(n)}{(n-1)^2}\cdot\tfrac{(n-1)(n+1)}{n^2} = \tfrac{n+1}{2n}$$

and so $\prod_{n=2}^{\infty}\left(1-\tfrac{1}{n^2}\right) = \lim_{l\to\infty} P_l = \lim_{l\to\infty}\tfrac{n+1}{2n} = \tfrac{1}{2}$.

(b) $\displaystyle\prod_{n=0}^{\infty}\left(1+\frac{1}{2^{2^n}}\right)$

Solution: Let $P_n = \displaystyle\prod_{k=0}^{n}\left(1+\frac{1}{2^{2^k}}\right)$. Then

$$\begin{aligned}\left(1-\tfrac{1}{2}\right)P_n &= \left(1-\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{2^2}\right)\left(1+\tfrac{1}{2^4}\right)\cdots\left(1+\tfrac{1}{2^{2^n}}\right) = \left(1-\tfrac{1}{2^2}\right)\left(1+\tfrac{1}{2^2}\right)\left(1+\tfrac{1}{2^4}\right)\cdots\left(1+\tfrac{1}{2^{2^n}}\right)\\ &= \left(1-\tfrac{1}{2^4}\right)\left(1+\tfrac{1}{2^4}\right)\left(1+\tfrac{1}{2^8}\right)\cdots\left(1+\tfrac{1}{2^{2^n}}\right) = \cdots = \left(1-\tfrac{1}{2^{2^n}}\right)\left(1+\tfrac{1}{2^{2^n}}\right) = \left(1-\tfrac{1}{2^{2^{n+1}}}\right).\end{aligned}$$

Thus $P_n = \dfrac{1-\frac{1}{2^{2^{n+1}}}}{1-\frac{1}{2}} \to \dfrac{1}{1-\frac{1}{2}} = 2$ as $n\to\infty$.

(c) $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$

Solution: Let $P_n = \displaystyle\prod_{k=2}^{n}\frac{k^3-1}{k^3+1}$. Then

$$\begin{aligned}P_n &= \prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} = \prod_{k=2}^{n}\frac{k-1}{k+1}\prod_{k=2}^{n}\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\\ &= \left(\tfrac{1}{3}\cdot\tfrac{2}{4}\cdot\tfrac{3}{5}\cdots\tfrac{k-2}{k}\cdot\tfrac{k-1}{k+1}\right)\left(\tfrac{7}{3}\cdot\tfrac{13}{7}\cdot\tfrac{21}{13}\cdots\tfrac{(k-1)^2+(k-1)+1}{(k-2)^2+(k-2)+1}\cdot\tfrac{k^2+k+1}{(k-1)^2+(k-1)+1}\right)\\ &= \left(\tfrac{1\cdot 2}{k(k+1)}\right)\left(\tfrac{k^2+k+1}{3}\right) \to \tfrac{2}{3}\text{ as } n\to\infty\,.\end{aligned}$$

6: (a) For $n \geq 1$, let $a_n = \binom{-2/3}{n}\left(-\frac{1}{2}\right)^n = \frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{6^n\, n!}$.

(i) Find the smallest $\ell \in \mathbb{Z}^+$ such that when the sum $S = \sum_{n=1}^{\infty}(-1)^n a_n$ is approximated by $S \cong S_\ell = \sum_{n=1}^{\ell}(-1)^n a_n$, the error is $\left|S - S_\ell\right| \leq \frac{1}{30}$.

Solution: Note that $a_{n+1} = a_n \cdot \dfrac{3n+2}{6(n+1)} < \frac{1}{2}\, a_n$ so $\{a_n\}$ decreases with limit 0, and so we can apply the A.S.T. The first few terms of $\{a_n\}$ are

$$a_1 = \tfrac{2}{6} = \tfrac{1}{3},\ a_2 = \tfrac{2\cdot 5}{6^2\cdot 2} = \tfrac{5}{36},\ a_3 = \tfrac{2\cdot 5\cdot 8}{6^3\cdot 3!} = \tfrac{5}{81},\ a_4 = \tfrac{2\cdot 5\cdot 8\cdot 11}{6^4\cdot 4!} = \tfrac{55}{1944},\ a_5 = \tfrac{2\cdot 5\cdot 8\cdot 11\cdot 14}{6^5\cdot 5!} = \tfrac{77}{5832}$$

By the A.S.T, if we approximate S by $S \cong S_3$ then the error is $|S - S_3| < a_4 = \frac{55}{1944} < \frac{55}{1650} = \frac{1}{30}$, while if we approximate S by $S \cong S_2$ then the error is $|S - S_2| > a_3 - a_4 = \frac{5}{81} - \frac{55}{1944} = \frac{65}{1944} > \frac{65}{1950} = \frac{1}{30}$.

(ii) Find the smallest $\ell \in \mathbb{Z}^+$ such that when the sum $T = \sum_{n=1}^{\infty} a_n$ is approximated by $T \cong T_\ell = \sum_{n=1}^{\ell} a_n$, the error is $T - T_\ell \leq \frac{1}{30}$.

Solution: Note that for $n \geq 5$ we have $a_n = a_5\, \frac{17\cdot 20\ldots(3n-1)}{6^{n-5} n!/5!} \leq a_5\, \frac{18\cdot 21\cdots(3n)}{6^{n-5} n!/5!} = a_5\, \frac{3^{n-5} n!/5!}{6^{n-5} n!/5!} = a_5\, \frac{1}{2^{n-5}}$, and so if we approximate T by $T \cong T_4$ then, using the C.T, the error is

$$T - T_4 = \sum_{n=5}^{\infty} a_n \leq a_5 \sum_{n=5}^{\infty} \tfrac{1}{2^{n-5}} = 2\, a_5 = \tfrac{77}{2916} < \tfrac{77}{2310} = \tfrac{1}{30}\,.$$

On the other hand, if we approximate T by $T \cong T_3$ then the error is

$$T - T_3 = \sum_{n=4}^{\infty} a_n > a_4 + a_5 = \tfrac{55\cdot 3 + 77}{5832} = \tfrac{121}{2916} > \tfrac{121}{3630} = \tfrac{1}{30}\,.$$

(b) Let $f(x) = \dfrac{1}{x(\ln x)^2}$, let $a_n = f(n)$ for $n \geq 2$, let $S = \sum_{n=2}^{\infty} a_n$, and let $S_\ell = \sum_{n=2}^{\ell} a_n$.

(i) Find a value of $\ell \in \mathbb{Z}^+$ such that if we approximate S by $S \cong S_\ell$ then the error is at most $\frac{1}{100}$.

Solution: Note that $f(x)$ is positive, continuous and decreasing for $x > 1$, so we can apply the I.T. If we approximate S by $S \cong S_l$ then by the I.T. the error is

$$E = S - S_l = \sum_{n=l+1}^{\infty} a_n \leq \int_l^\infty f(x)\, dx = \int_l^\infty \frac{dx}{x(\ln x)^2} = \left[\frac{-1}{\ln x}\right]_l^\infty = \frac{1}{\ln l}\,.$$

To get $E \leq \frac{1}{100}$ we can choose l so that $\frac{1}{\ln l} \geq \frac{1}{100}$, that is $\ln l \geq 100$, so we can take $l \geq e^{100}$ (a huge number).

(ii) Use a calculator to find a value of $\ell \in \mathbb{Z}^+$ such that if we approximate S by

$$S \cong S_\ell + \tfrac{1}{2}\left(\int_\ell^\infty f(x)\, dx + \int_{\ell+1}^\infty f(x)\, dx\right)$$

then the error is at most $\frac{1}{100}$.

Solution: By the I.T, if we make the above approximation then the error is

$$E \leq \tfrac{1}{2}\left(\int_l^\infty f(x)\, dx - \int_{l+1}^\infty f(x)\, dx\right) = \tfrac{1}{2}\left(\frac{1}{\ln l} - \frac{1}{\ln(l+1)}\right),$$

so to get $E \leq \frac{1}{100}$ we can choose l so that $\frac{1}{2}\left(\frac{1}{\ln l} - \frac{1}{\ln(l+1)}\right) \leq \frac{1}{100}$. By trial and error with the help of a calculator, we find that $\frac{1}{2}\left(\frac{1}{\ln(10)} - \frac{1}{\ln(11)}\right) \cong 0.0086 < \frac{1}{100}$ and so we can take $l = 10$.

7: Determine, with proof, which of the following statements are true.

(a) If $\sum a_n$ converges that $\sum e^{a_n}$ diverges.

Solution: This is true, and we give a proof. Suppose that $\sum a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$ by the D.T, and so $\lim_{n\to\infty} e^{a_n} = e^0 = 1$. Since $\lim_{n\to\infty} e^{a_n} \neq 0$, $\sum e^{a_n}$ diverges by the D.T.

(b) If $\sum a_n$ converges then $\sum {a_n}^2$ converges.

Solution: This is false, and we provide a counterexample. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum a_n$ converges by the A.S.T, but $\sum {a_n}^2 = \sum \frac{1}{n}$ which diverges.

(c) If $\sum a_n$ converges and $\sum |b_n|$ converges, then $\sum a_n b_n$ converges.

Solution: This is true. Indeed, suppose that $\sum a_n$ converges and that $\sum |b_n|$ converges. Since $\sum a_n$ converges, we have $a_n \to 0$ (by the Divergence Test) so we can choose N so that $n \geq N \implies |a_n| \leq 1$. Then for $n \geq N$ we have $0 \leq |a_n b_n| = |a_n||b_n| \leq |b_n|$, and so, since $\sum |b_n|$ converges, $\sum |a_n b_n|$ also converges by the Comparison Test. Since absolute convergence implies convergence, $\sum a_n b_n$ converges, too.

(d) If $f(x)$ is positive and continuous and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^\infty f(n)$ converges.

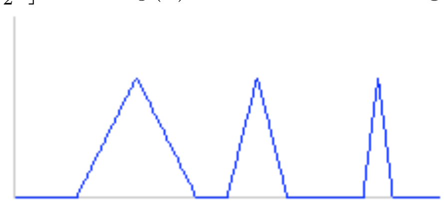
Solution: This is false, and we provide a counterexample. Let

$$g_1(x) = \begin{cases} 2x-1 \text{ if } \frac{1}{2} \leq x \leq 1, \\ 3-2x \text{ if } 1 \leq x \leq \frac{3}{2}, \\ 0 \quad \text{otherwise}, \end{cases} \quad g_2(x) = \begin{cases} 4x-7 \text{ if } \frac{7}{4} \leq x \leq 2, \\ 9-4x \text{ if } 2 \leq x \leq \frac{9}{4}, \\ 0 \quad \text{otherwise}, \end{cases} \quad g_3(x) = \begin{cases} 8x-23 \text{ if } \frac{23}{8} \leq x \leq 3, \\ 25-8x \text{ if } 3 \leq x \leq \frac{25}{8}, \\ 0 \quad \text{otherwise}, \end{cases}$$

and in general, for $k \geq 1$ let

$$g_k(x) = \begin{cases} 2^k x - k2^k + 1 \text{ if } k - \frac{1}{2^k} \leq x \leq k, \\ k2^k + 1 - 2^k x \text{ if } k \leq x \leq k + \frac{1}{2^k}, \\ 0 \quad \text{otherwise}. \end{cases}$$

Then $\int_0^\infty g_1(x)\,dx = \frac{1}{2}$, $\int_0^\infty g_2(x)\,dx = \frac{1}{4}$, $\int_0^\infty g_3(x)\,dx = \frac{1}{8}$, and in general $\int_0^\infty g_k(x)\,dx = \frac{1}{2^k}$. Now let $g(x) = g_k(x)$ when $x \in \left[k - \frac{1}{2^k}, k + \frac{1}{2^k}\right]$ and let $g(x) = 0$ otherwise. The graph of $g(x)$ is shown below.



Then $g(x)$ is nonnegative and continuous, and $\int_1^\infty g(x)\,dx$ converges, indeed $\int_0^\infty g(x)\,dx = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. On the other hand we have $g(n) = 1$ for all integers $n \geq 1$, so $\sum_{n=1}^\infty g(n) = \infty$. For a strictly positive counterexample, let $f(x) = g(x) + e^{-x}$.

(e) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$ then $\left(\sum a_n \text{ converges } \iff \sum b_n \text{ converges}\right)$.

Solution: This is false. For a counterexample, let $a_{2n} = \frac{1}{\sqrt{n}}$ and $a_{2n+1} = -\frac{1}{\sqrt{n}}$ for all $n \geq 1$, so we have $\{a_n\} = \left\{1, -1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \cdots\right\}$, and let $b_{2n} = \left(1 + \frac{1}{\sqrt{n}}\right) a_{2n} = \left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right)$ and $b_{2n+1} = a_{2n+1} = -\frac{1}{\sqrt{n}}$. Note that $\sum a_n$ converges by the A.S.T. Also, we have $\dfrac{a_{2n+1}}{b_{2n+1}} = 1$ for all n and $\dfrac{a_{2n}}{b_{2n}} = \dfrac{1}{1 + \frac{1}{\sqrt{n}}} \to 1$ as $n \to \infty$, and so $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$. But $\sum b_n$ diverges, since, writing S_l for the l^{th} partial sum of $\sum\limits_{n=1}^{\infty} b_n$, we have $S_{2l+1} = \sum\limits_{n=1}^{l} (a_{2n} + a_{2n+1}) = \sum\limits_{n=1}^{l} \left(\left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right) - \frac{1}{\sqrt{n}}\right) = \sum\limits_{n=1}^{l} \frac{1}{n} \to \infty$ as $l \to \infty$.

(f) If $\sum a_n$ converges then $\sum \frac{a_n}{1+a_n}$ converges.

Solution: This is false, and we provide a counterexample. Note that $\frac{\frac{1}{m}}{1+\frac{1}{m}} = \frac{1}{m+1}$ and $\frac{-\frac{1}{m}}{1+\left(-\frac{1}{m}\right)} = \frac{-1}{m-1}$, and that $\frac{1}{m+1} - \frac{1}{m-1} = \frac{-2}{m^2-1}$. Thus for a counterexample, we can let

$$a_n = \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{3}, \cdots, \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{4}, \cdots, \tfrac{1}{4}, -\tfrac{1}{4}, \cdots$$

where for each m the pair $\frac{1}{m}, -\frac{1}{m}$ is repeated $m^2 - 1$ times so that $\frac{1}{m+1} - \frac{1}{m-1} + \cdots + \frac{1}{m+1} - \frac{1}{m-1} = -2$. Note that $\sum a_n$ converges by the A.S.T, but $\sum \dfrac{a_n}{1 + a_n} = -\infty$.

8: Let $(a_n)_{n\geq 1}$ be a sequence with $a_n \geq 0$ for all $n \geq 1$, and let $S_n = \sum\limits_{k=1}^{n} a_k$.

(a) Show that if $\{a_n\}_{n\geq 1}$ is decreasing and $\sum\limits_{n\geq 1} a_n$ converges, then $\lim\limits_{n\to\infty} n a_n = 0$.

Solution: Suppose that $\{a_n\}_{n\geq 1}$ is decreasing and $\displaystyle\sum_{n\geq 1} a_n$ converges. Let $\epsilon > 0$. By the Cauchy Criterion for convergence we can choose $N \geq 1$ so that

$$m > l > N \Longrightarrow \sum_{n=l+1}^{m} a_n < \tfrac{\epsilon}{2}\,.$$

Fix $l > N$. Since $a_n \to 0$ (by the Divergence Test) we can choose $M \geq l$ so that

$$m > M \Longrightarrow a_m < \tfrac{\epsilon}{2l}\,.$$

Since $\{a_n\}$ is decreasing so that $a_m \leq a_n$ for all $n \leq m$, for $m > M$ we have

$$m a_m = l a_m + \sum_{n=l+1}^{m} a_m \leq l a_m + \sum_{n=l+1}^{m} a_n < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Thus $\lim\limits_{m\to\infty} m a_m = 0$.

(b) Show that if $a_1 > 0$ and $\sum\limits_{n\geq 1} a_n$ diverges, then $\sum\limits_{n\geq 1} \frac{a_n}{S_n}$ also diverges.

Solution: Let $T_l = \sum\limits_{n=1}^{l} \frac{a_n}{S_n}$. Let $l_1 = 1$. Note that $T_{l_1} = T_1 = \frac{a_1}{S_1} = \frac{a_1}{a_1} = 1 > \frac{1}{2}$. Suppose, inductively, that we have found l_k so that $T_{l_k} > \frac{k}{2}$. Since $\sum\limits_{n=l_k+1}^{\infty} a_n = \infty$, we can choose l_{k+1} so that $\sum\limits_{n=l_k+1}^{l_{k+1}} a_n > S_{l_k}$, that is $S_{l_{k+1}} - S_{l_k} > S_{l_k}$. Then, since $\{S_n\}$ is increasing so that $S_n \leq S_{l_{k+1}}$ for all $n \leq l_{k+1}$, we have

$$\begin{aligned} T_{l_{k+1}} &= T_{l_k} + \sum_{n=l_k+1}^{l_{k+1}} \frac{a_n}{S_n} \geq T_{l_k} + \sum_{n=l_k+1}^{l_{k+1}} \frac{a_n}{S_{l_{k+1}}} = T_{l_k} + \frac{S_{l_{k+1}} - S_{l_k}}{S_{l_k+1}} \\ &= T_{l_k} + \frac{S_{l_{k+1}} - S_{l_k}}{S_{l_k} + (S_{l_{k+1}} - S_{l_k})} > T_{l_k} + \frac{S_{l_{k+1}} - S_{l_k}}{2(S_{l_{k+1}} - S_{l_k})} > T_{l_k} + \tfrac{1}{2} > \tfrac{k}{2} + \tfrac{1}{2} = \tfrac{k+1}{2}\,. \end{aligned}$$

Thus for all $k \geq 1$ we can find l_k such that $T_{l_k} > \frac{k}{2}$. Since $\{T_l\}$ is increasing, we obtain

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n} = \lim_{l\to\infty} T_l = \infty\,.$$

9: Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. When $f:\mathbb{Z}^+\to\mathbb{Z}^+$ is a bijective map and we write $n_k=f(k)$, the sequence $(a_{n_k})_{k\geq 1}$ is called a **rearrangement** of the sequence $(a_n)_{n\geq 1}$. Prove each of the following.

(a) If $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, then $\sum\limits_{k=1}^{\infty} a_{n_k}=\sum\limits_{n=1}^{\infty} a_n$ for every rearrangement $(a_{n_k})_{k\geq 1}$.

Solution: I may include a solution later.

(b) If $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally then for every $r\in\mathbb{R}$ there is a rearrangement $(a_{n_k})_{k\geq 1}$ such that $\sum\limits_{k=1}^{\infty} a_{n_k}=r$.

Solution: I may include a solution later.

(c) If $\sum\limits_{n=1}^{\infty} a_n$ diverges with $\sum\limits_{n=1}^{\infty} a_n\neq-\infty$ then there is a rearrangement $(a_{n_k})_{k\geq 1}$ such that $\sum\limits_{k=1}^{\infty} a_{n_k}=\infty$.

Solution: We sketch a proof. Suppose that $\sum a_n$ diverges and that $\sum a_n\neq-\infty$. Let p_n be the n^{th} non-negative term in $\{a_n\}$ and let q_n be the n^{th} negative term in $\{a_n\}$. Since each p_n is non-negative, either $\sum p_n$ converges or $\sum p_n=\infty$. Since each q_n is negative, either $\sum q_n$ converges or $\sum q_n=-\infty$. Note that it is not possible for $\sum p_n$ to converge, since if $\sum p_n$ converges and $\sum q_n$ converges then $\sum a_n$ also converges (absolutely), and if $\sum p_n$ converges and $\sum q_n=-\infty$ then $\sum a_n=-\infty$. Thus we must have $\sum p_n=\infty$.

Let P_l and Q_l denote the l^{th} partial sums for $\sum\limits_{n=1}^{\infty} p_n$ and $\sum\limits_{n=1}^{\infty} q_n$. Let k_1 be the smallest positive integer such that $P_{k_1}>1+|Q_1|=1-Q_1$. Having chosen $k_1,\cdots,k_{n-1}$, let k_n be the smallest integer with $k_n>k_{n-1}$ such that $P_{k_n}>n-Q_n$ (each k_n exists since $\sum p_n=\infty$). Let $\{b_n\}$ be the rearrangement of $\{a_n\}$ given by

$$\{b_n\}=\big\{p_1,\cdots,p_{k_1},q_1,p_{k_1+1},\cdots,p_{k_2},q_2,p_{k_2+1},\cdots,p_{k_3},q_3,\cdots\big\}$$

Verify that $\sum b_n=\infty$.