MATH 148 Calculus 2, Exercises for Chapter 6

1: (a) Let $a_1=6$ and for $n\ge 1$ let $a_{n+1}=1+2^{a_n/3}$. Determine whether (a_n) converges, and if so find the limit.

(b) Let $a_1=\frac{7}{2}$ and for $n\ge 1$ let $a_{n+1}=\dfrac{6}{5-a_n}$. Determine whether (a_n) converges, and if so find the limit.

(c) Let $(x_k)_{k\ge 0}$ be a sequence in $\mathbb{R}$ with $\left|x_k-x_{k-1}\right|\le\frac{1}{k^2}$ for all $k\ge 1$. Show that (x_k) converges in $\mathbb{R}$.

2: Determine which of the following series converge.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{\sqrt{n}}{2n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,2^{1/n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!n^n}{(2n)!}$ (d) $\displaystyle\sum_{n=1}^{\infty}\Big(n\sin^{-1}\left(\tfrac{1}{n}\right)-1\Big)$.

3: Find the sum of each of the following series, if the sum exists.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{n+1}+3^n}{6^{n-1}}$ (b) $\displaystyle\sum_{n=3}^{\infty}\frac{2}{n^2-4}$ (c) $\displaystyle\sum_{n=-1}^{\infty}e^{-(n\ln 2)/2}$ (d) $\displaystyle\sum_{n=2}^{\infty}\frac{6n^2}{n^6-1}$

4: Find the sum of each of the following series, if the sum exists.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n+1)!}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ (c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{a_{n-1}a_{n+1}}$, where $\{a_n\}$ is the Fibonacci sequence.

5: Given a sequence $(a_n)_{n\ge k}$, we define the infinite product $\displaystyle\prod_{n=k}^{\infty}a_n$ to be $\displaystyle\lim_{\ell\to\infty}P_\ell$ where $P_\ell=\displaystyle\prod_{n=k}^{\ell}a_n$, if the limit exists. Evaluate each of the following infinite products.

(a) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (b) $\displaystyle\prod_{n=0}^{\infty}\left(1+\frac{1}{2^{2^n}}\right)$ (c) $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$

6: (a) For $n\ge 1$, let $a_n=\binom{-2/3}{n}\left(-\frac{1}{2}\right)^n=\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{6^n\,n!}$.

(i) Find the smallest $\ell\in\mathbb{Z}^+$ such that when the sum $S=\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ is approximated by $S\cong S_\ell=\displaystyle\sum_{n=1}^{\ell}(-1)^n a_n$, the error is $\left|S-S_\ell\right|\le\frac{1}{30}$.

(ii) Find the smallest $\ell\in\mathbb{Z}^+$ such that when the sum $T=\displaystyle\sum_{n=1}^{\infty}a_n$ is approximated by $T\cong T_\ell=\displaystyle\sum_{n=1}^{\ell}a_n$, the error is $T-T_\ell\le\frac{1}{30}$.

(b) Let $f(x)=\dfrac{1}{x(\ln x)^2}$, let $a_n=f(n)$ for $n\ge 2$, let $S=\displaystyle\sum_{n=2}^{\infty}a_n$, and let $S_\ell=\displaystyle\sum_{n=2}^{l}a_n$.

(i) Find a value of $\ell\in\mathbb{Z}^+$ such that if we approximate S by $S\cong S_\ell$ then the error is at most $\frac{1}{100}$.

(ii) Use a calculator to find a value of $\ell\in\mathbb{Z}^+$ such that if we approximate S by

$$S\cong S_\ell+\tfrac{1}{2}\left(\int_\ell^\infty f(x)\,dx+\int_{\ell+1}^\infty f(x)\,dx\right)$$

then the error is at most $\frac{1}{100}$.

7: Determine, with proof, which of the following statements are true.

(a) If $\sum a_n$ converges that $\sum e^{a_n}$ diverges.

(b) If $\sum a_n$ converges then $\sum {a_n}^2$ converges.

(c) If $\sum a_n$ converges and $\sum|b_n|$ converges, then $\sum a_nb_n$ converges.

(d) If $f(x)$ is positive and continuous and $\displaystyle\int_1^\infty f(x)\,dx$ converges then $\displaystyle\sum_{n=1}^{\infty}f(n)$ converges.

(e) If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=1$ then $\left(\sum a_n\text{ converges}\iff\sum b_n\text{ converges}\right)$.

(f) If $\sum a_n$ converges then $\sum\frac{a_n}{1+a_n}$ converges.

8: Let $(a_n)_{n\geq 1}$ be a sequence of real numbers with $a_n \geq 0$ for all $n \geq 1$, and let $S_n = \sum_{k=1}^{\ell} a_k$.

(a) Show that if $(a_n)_{n\geq 1}$ is decreasing and $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

(b) Show that if $a_1 > 0$ and $\sum_{n\geq 1} a_n$ diverges, then $\sum_{n\geq 1} \frac{a_n}{S_n}$ also diverges.

9: Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. When $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a bijective map and we write $n_k = f(k)$, the sequence $(a_{n_k})_{k\geq 1}$ is called a **rearrangement** of the sequence $(a_n)_{n\geq 1}$. Prove each of the following.

(a) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{k=1}^{\infty} a_{n_k} = \sum_{n=1}^{\infty} a_n$ for every rearrangement $(a_{n_k})_{k\geq 1}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges conditionally then for every $r \in \mathbb{R}$ there is a rearrangement $(a_{n_k})_{k\geq 1}$ such that $\sum_{k=1}^{\infty} a_{n_k} = r$.

(c) If $\sum_{n=1}^{\infty} a_n$ diverges with $\sum_{n=1}^{\infty} a_n \neq -\infty$ then there is a rearrangement $(a_{n_k})_{k\geq 1}$ such that $\sum_{k=1}^{\infty} a_{n_k} = \infty$.