MATH 148 Calculus 2, Solutions to the Exercises for Chapter 5

1: (a) Verify that $y = x\sin x$ is a solution of the DE $y\,(y'' + y) = x\sin 2x$.

Solution: We have $y' = \sin x + x\cos x$ and $y'' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$ and so

$$\begin{aligned} y\,(y''+y) &= (x\sin x)(2\cos x - x\sin x + x\sin x) \\ &= (x\sin x)(2\cos x) \\ &= x(2\sin x\cos x) \\ &= x\sin 2x\,. \end{aligned}$$

(b) Find all the solutions of the form $y = ax^2 + bx + c$ to the DE $\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1$.

Solution: For $y = ax^2 + bx + c$ we have $y' = 2ax + b$, so

$$\begin{aligned} \big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1 &\iff \big(y'(x)\big)^2 + 4x - 3y(x) - x^2 - 1 = 0 \\ &\iff (2ax+b)^2 + 4x - 3(ax^2+bx+c) - x^2 - 1 = 0 \\ &\iff (4a^2 - 3a - 1)x^2 + (4ab + 4 - 3b)x + (b^2 - 3c - 1) = 0 \\ &\iff 4a^2 - 3a - 1 = 0\ ,\ 4ab + 4 = 3b\ ,\ \text{and } b^2 = 3c + 1 \end{aligned}$$

From $4a^2 - 3a - 1 = 0$ we get $(4a+1)(a-1) = 0$ and so $a = -\frac{1}{4}$ or $a = 1$. When $= -\frac{1}{4}$, the equation $4ab + 4 = 3b$ gives $-1 + 4 = 3b$ so $b = 1$, and then the equation $b^2 = 3c + 1$ gives $1 = 3c + 1$ so $c = 0$. When $a = 1$, $4ab + 4 = 3b$ gives $4b + 4 = 3b$ so $b = -4$ and then $b^2 = 3c + 1$ gives $16 = 3c + 1$ so $c = 5$. Thus there are two solutions, and they are $y = -\frac{1}{4}x^2 + x$ and $y = x^2 - 4x + 5$.

(c) Find constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

Solution: Let $y = e^{rx}$. Then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$ and so $y'' + 3y' + 2y = 0 \iff r^2\,e^{rx} + 3r\,e^{rx} + 2\,e^{rx} = 0$ $\iff (r^2 + 3r + 2)e^{rx} = 0 \iff (r+1)(r+2)\,e^{rx} = 0 \iff r = -1$ or $r = -2$. Thus we can take $r_1 = -1$ and $r_2 = -2$.

Now, let $y = a\,e^{r_1 x} + b\,e^{r_2 x} = a\,e^{-x} + b\,e^{-2x}$. Then $y' = -a\,e^{-x} - 2b\,e^{-2x}$ and $y'' = a\,e^{-x} + 4b\,e^{-2x}$ and so we have $y'' + 3y' + 2y = a\,e^{-x} + 4b\,e^{-2x} - 3a\,e^{-x} - 6b\,e^{-2x} + 2a\,e^{-x} + 2b\,e^{-2x} = 0$. This shows that $y = a\,e^{-x} + b\,e^{-2x}$ is a solution to the DE. Also, note that $y(0) = a + b$ and $y'(0) = -a - 2b$, and so to get $y(0) = 1$ and $y'(0) = 0$ we need $a + b = 1$ and $-a - 2b = 0$. Solve these two equations to get $a = 2$ and $b = -1$. Thus the required solution is $y = 2\,e^{-x} - e^{-2x}$.

2: Find the general solution to each of the following DEs.

(a) $x\,y' + y = \sqrt{x}$

Solution: This DE is linear since we can write it in the form $y' + \frac{1}{x}\,y = x^{-1/2}$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and so the solution is $y = \frac{1}{x}\int x \cdot x^{-1/2}\,dx = \frac{1}{x}\int x^{1/2}\,dx = \frac{1}{x}\left(\frac{2}{3}\,x^{3/2} + c\right) = \frac{2}{3}\sqrt{x} + \dfrac{c}{x}$.

(b) $\sqrt{x}\,y' = 1 + y^2$

Solution: This DE is separable. We can write it as $\dfrac{dy}{1+y^2} = x^{-1/2}\,dx$ and then integrate both sides to get $\tan^{-1} y = 2x^{1/2} + c$, that is $y = \tan\left(2\sqrt{x} + c\right)$.

(c) $y' = 2xy^2 + y^2 + 8x + 4$

Solution: This DE is separable since we can write it as $y' = (2x+1)(y^2+4)$ or as $\dfrac{dy}{y^2+4} = (2x+1)dx$. Integrate both sides to get

$$\begin{aligned}\int \frac{dy}{y^2+4} &= \int 2x+1\;dx\\ \tfrac{1}{2}\tan^{-1}(y/2) &= x^2 + x + c\\ y &= 2\tan\left(2(x^2+x+c)\right).\end{aligned}$$

(d) $y' + y\tan x = \sin^2 x$

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x = \dfrac{1}{\cos x}$ and the solution is

$$\begin{aligned}y &= \cos x \int \frac{\sin^2 x}{\cos x}\,dx = \cos x \int \frac{1-\cos^2 x}{\cos x}\,dx = \cos x \int \sec x - \cos x\;dx\\ &= \cos x\Big(\ln\big|\sec x + \tan x\big| - \sin x + c\Big).\end{aligned}$$

3: Find the solution to each of the following IVPs.

(a) $x\,y' = y^2 + y$ with $y(1) = 1$.

Solution: This DE is separable. We write it as $\dfrac{dy}{y^2+y} = \dfrac{dx}{x}$. Integrate both sides, using partial fractions for the integral on the left, to get

$$\begin{aligned} \int \frac{1}{y} - \frac{1}{y+1}\, dy &= \int \frac{1}{x}\, dx \\ \ln y - \ln(y+1) &= \ln x + c \\ \ln\left(\frac{y}{y+1}\right) &= \ln x + c \\ \frac{y}{y+1} &= e^{\ln x + c} = a\,x\,, \end{aligned}$$

where $a = \ln c$. Put in $y(1) = 1$ to get $\frac{1}{2}$, so we have $\dfrac{y}{y+1} = \dfrac{x}{2}$ so $2y = x(y+1) = xy + x$, that is $y(2-x) = x$, so the solution is $y = \dfrac{x}{2-x}$.

(b) $x\,y' + 2y = \ln x$ with $y(1) = 0$.

Solution: This DE is linear since we can write it as $y' + \frac{2}{x}\,y = \frac{1}{x}\ln x$. An integrating factor is given by $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and so the solution is $y = \dfrac{1}{x^2}\displaystyle\int x\,\ln x\, dx$. We integrate by parts using $u = \ln x$ and $dv = x\,dx$ so that $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}\,x^2$ to get

$$\begin{aligned} y &= \frac{1}{x^2}\int x\,\ln x\, dx \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}\,x\,dx\right) \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}\,x^2\ln x - \tfrac{1}{4}\,x^2 + c\right) \\ &= \frac{c}{x^2} + \tfrac{1}{2}\,\ln x - \tfrac{1}{4} \end{aligned}$$

Put in $y(1) = 0$ to get $0 = c - \frac{1}{4}$, so we have $c = \frac{1}{4}$ and the solution is $y = \frac{1}{4}\left(\dfrac{1}{x^2} + 2\ln x - 1\right)$.

(c) $y' + xy = x^3$ with $y(0) = 1$.

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int x\,dx} = e^{\frac{1}{2}x^2}$. The solution to the DE is

$$y = e^{-\frac{1}{2}x^2}\int x^3 e^{\frac{1}{2}x^2}\, dx\,.$$

Integrate by parts using $u = x^2$, $du = 2x\,dx$, $v = e^{\frac{1}{2}x^2}$, $dv = xe^{\frac{1}{2}x^2}$ to get

$$y = e^{-\frac{1}{2}x^2}\left(x^2 e^{\frac{1}{2}x^2} - \int 2xe^{\frac{1}{2}x^2}\, dx\right) = e^{-\frac{1}{2}x^2}\left(x^2e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2} + c\right) = x^2 - 2 + ce^{-\frac{1}{2}x^2}\,.$$

To get $y(0) = 1$ we need $-2 + c = 1$ so $c = 3$. Thus the solution to the IVP is

$$y = x^2 - 2 + 3e^{-\frac{1}{2}x^2} \text{ for all } x\,.$$

(d) $y' = \dfrac{y}{x+y^2}$ with $y(3) = 1$.

Solution: We interchange the rolls of x and y, and solve this DE for $x = x(y)$. We have

$$x'(y) = \frac{1}{y'(x)} = \frac{x+y^2}{y}$$

This DE is linear since we can write it as $x' - \frac{1}{y}\,x = y$. An integrating factor is $\lambda = e^{\int -\frac{1}{y}\,dy} = e^{-\ln y} = \frac{1}{y}$ and the solution is

$$x = y \int 1\,dy = y(y+c)\,.$$

To get $y(3) = 1$ (that is to get $x(1) = 3$) we need $2 = 1 + c$ so $c = 2$, and so the solution is

$$x = y(y+2) = (y+1)^2 - 1\,.$$

Solve this for $y = y(x)$ to get $y = -1 \pm \sqrt{x+1}$. Note that to satisfy $y(3) = 1$ we need to use the + sign, so

$$y = -1 + \sqrt{x+1}\,.$$

4: Consider the IVP $y' = 2(x+y) - \frac{1}{2}$ with $y(0) = 0$.

(a) Find the exact solution $y = f(x)$ to the above IVP.

Solution: The DE is linear as we can write it as $y' - 2y = 2x - \frac{1}{2}$. An integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$, and the solution is $y = e^{2x} \int \left(2x - \frac{1}{2}\right) e^{-2x}\,dx$. Integrate by parts using $u = 2x - \frac{1}{2}$, $du = 2\,dx$, $v = \frac{-1}{2}e^{-2x}$ and $dv = e^{-2x}\,dx$ to get

$$y = e^{2x}\left(\left(-x + \tfrac{1}{4}\right)e^{-2x} + \int e^{-2x}\,dx\right) = e^{2x}\left(\left(-x+\tfrac{1}{4}\right)e^{-2x} - \tfrac{1}{2}e^{-2x} + c\right) = ce^{2x} - \left(x + \tfrac{1}{4}\right).$$

To get $y(0) = 0$ we need $0 = c - \frac{1}{4}$ so $c = \frac{1}{4}$, and the solution is

$$y = \tfrac{1}{4}e^{2x} - \left(x + \tfrac{1}{4}\right).$$

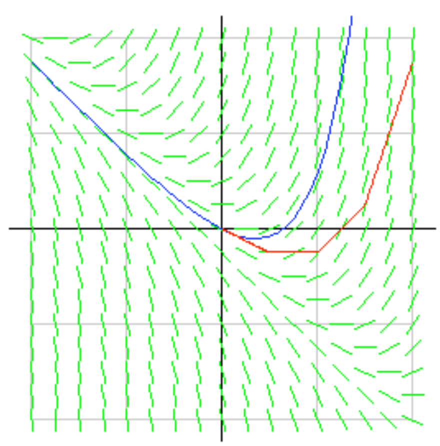
(b) Apply Euler's method with step size $\Delta x = \frac{1}{2}$ to find a polygonal approximation $y = g(x)$ for $0 \le x \le 2$ to the above solution $y = f(x)$.

Solution: We make a table showing the values of x_k, y_k and m_k, where $x_0 = 0$, $y_0 = 0$, $m_k = 2(x_k + y_k) - \frac{1}{2}$, $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + m_k \Delta x$.

k	x_k	y_k	m_k
0	0	0	$2(0+0) - \frac{1}{2} = -\frac{1}{2}$
1	$\frac{1}{2}$	$0 + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$	$2\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2} = 0$
2	1	$-\frac{1}{4} + (0)\left(\frac{1}{2}\right) = -\frac{1}{4}$	$2\left(1 - \frac{1}{4}\right) - \frac{1}{2} = 1$
3	$\frac{3}{2}$	$-\frac{1}{4} + (1)\left(\frac{1}{2}\right) = \frac{1}{4}$	$2\left(\frac{3}{2} + \frac{1}{4}\right) - \frac{1}{2} = 3$
4	2	$\frac{1}{4} + (3)\left(\frac{1}{2}\right) = \frac{7}{4}$	

(c) Sketch the direction field for the given DE along with the graph of the exact solution $y = f(x)$ and the graph of the polygonal solution $y = g(x)$.

Solution: The direction field is shown in green, the exact solution is in blue, and the polygonal approximation is in red.



(d) Let $g_n(x)$ be the polygonal approximation to the solution $y = f(x)$ obtained by applying Euler's method with step size $\Delta x = \frac{1}{n}$. Show that $\lim\limits_{n\to\infty} g_n(1) = f(1)$.

Solution: From part (a) we have $f(1) = \frac{1}{4}e^2 - \frac{5}{4}$. Fix n and let x_k, y_k and m_k be obtained by applying Euler's method with step size $\Delta x = \frac{1}{n}$. Note that $g_n(1) = y_n$. Since $x_0 = 0$ and $x_{k+1} = x_k + \Delta x = x_k + \frac{1}{n}$, we have $x_k = \frac{k}{n}$ for all $k \geq 0$. Since $y_{k+1} = y_k + m_k \Delta x$ with $m_k = 2(x_k + y_k) - \frac{1}{2} = 2\left(\frac{k}{n} + y_k\right) - \frac{1}{2}$ we have

$$y_{k+1} = y_k + \left(2\left(\frac{k}{n} + y_k\right) - \frac{1}{2}\right)\left(\frac{1}{n}\right) = y_k + \frac{2k}{n^2} + \frac{2}{n}\, y_k - \frac{1}{2n} = y_k\left(1 + \frac{2}{k}\right) + \frac{2k}{n^2} - \frac{1}{2n}\,.$$

Now let $z_k = y_k + \left(x_k + \frac{1}{4}\right) = y_k + \frac{k}{n} + \frac{1}{4}$ for all $k \geq 0$. Then $z_0 = \frac{1}{4}$ and

$$\begin{aligned} z_{k+1} &= y_{k+1} + \tfrac{k+1}{n} + \tfrac{1}{4} = y_k\left(1 + \tfrac{2}{n}\right) + \tfrac{2k}{n^2} - \tfrac{1}{2n} + \tfrac{k+1}{n} + \tfrac{1}{4} \\ &= \left(z_k - \left(\tfrac{k}{n} + \tfrac{1}{4}\right)\right)\left(1 + \tfrac{2}{n}\right) + \tfrac{2k}{n^2} + \tfrac{1}{2n} + \tfrac{k}{n} + \tfrac{1}{4} \\ &= z_k\left(1 + \tfrac{2}{n}\right)\,, \end{aligned}$$

and so $z_k = \frac{1}{4}\left(1 + \frac{2}{n}\right)^k$ for all $k \geq 0$. Since $y_k = z_k - \left(\frac{k}{n} + \frac{1}{4}\right)$, we have

$$y_k = \frac{1}{4}\left(1 + \frac{2}{n}\right)^k - \left(\frac{k}{n} + \frac{1}{4}\right) \text{ for all } k \geq 0\,.$$

In particular,

$$g_n(1) = y_n = \frac{1}{4}\left(1 + \frac{2}{n}\right)^n - \frac{5}{4}\,.$$

Since $\left(1 + \frac{2}{n}\right)^n = e^{n\ln\left(1 + \frac{2}{n}\right)}$ and, using l'Hôpital's Rule,

$$\lim_{n\to\infty} n\ln\left(1 + \frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{2}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-\frac{2}{n^2}}{1 + \frac{2}{n}}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2}{1 + \frac{2}{n}} = 2\,,$$

we have $\lim\limits_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = e^2$, and so $\lim\limits_{n\to\infty} g_n(1) = \frac{1}{4}\,e^2 - \frac{5}{4}$, as required.

5: (a) The amount $A(t)$ of a radioactive substance satisfies the DE

$$A'(t) = k\,A(t)$$

for some constant $k < 0$. The substance has a half-life of 10 seconds, which means that $A(10) = \frac{1}{2}\,A(0)$. If $A(5) = 100$ then find the exact time t at which $A(t) = 20$.

Solution: This DE is linear, since we can write it in the form $A' + kA = 0$. An integrating factor is $\lambda = e^{\int k\,dt} = e^{k\,t}$ and the general solution is $A(t) = e^{-k\,t}\int 0\,dt = c\,e^{-kt}$. Note that $A(0) = c$, so c is the initial amount. Since the half-life is 10, we have

$$A(10) = \tfrac{1}{2}\,c \Longrightarrow c\,e^{-10k} = \tfrac{1}{2}\,c \Longrightarrow e^{-10k} = \tfrac{1}{2} \Longrightarrow e^{10k} = 2 \Longrightarrow 10k = \ln 2 \Longrightarrow k = \tfrac{1}{10}\ln 2\,.$$

and so $A(t) = c\,e^{-(t/10)\ln 2} = c\,2^{-t/10}$. Also, we have $A(5) = 100 \Longrightarrow c\,2^{-1/2} = 100 \Longrightarrow c = 100\sqrt{2}$, and so $A(t) = \big(100\sqrt{2}\big)\,2^{-t/10}$. Finally, we have

$$\begin{aligned} A(t) = 20 &\iff \big(100\sqrt{2}\big)\,2^{-t/10} = 20 \\ &\iff 2^{t/10} = \frac{100\sqrt{2}}{20} = 5\sqrt{2} = \sqrt{50} \\ &\iff \frac{t}{10} = \log_2\sqrt{50} = \tfrac{1}{2}\log_2 50 \\ &\iff t = 5\log_2 50\,. \end{aligned}$$

(b) A pot of boiling water is removed from the heat and placed on a table in a room. The temperature $T(t)$ of the water at time t satisfies **Newton's Law of Cooling**, that is

$$T'(t) = k\big(C - T(t)\big)$$

for some constant $k > 0$, where C is the room temperature. After 2 minutes, the water has cooled from 100° to 84°. After another 2 minutes, it has cooled to 72°. What is the temperature in the room?

Solution: This DE is linear since we can write it in the form $T' + kT = kC$. An integrating factor is $\lambda = e^{\int k\,dt} = e^{kt}$, and the general solution is $T(t) = e^{-kt}\int kC\,e^{kt} = e^{-kt}\big(Ce^{kt} + a\big) = C + a\,e^{-kt}$. Since $T(0) = 100$ we have $C + a = 100$, so $a = 100 - C$, and so

$$T(t) = C + (100 - C)\,e^{-kt}\,.$$

Since $T(2) = 84$, we have $C + (100 - C)\,e^{-2k} = 84$, and so $e^{-2k} = \dfrac{84 - C}{100 - C}$, and since $T(4) = 72$, we have $C + (100 - C)\,e^{-4k} = 72$, and so $e^{-4k} = \dfrac{72 - C}{100 - C}$. Using the fact that $e^{-4k} = \big(e^{-2k}\big)^2$, we have $\dfrac{72 - C}{100 - C} = \dfrac{(84 - C)^2}{(100 - C)^2}$. This gives $(72-C)(100-C) = (84-C)^2$, that is $7200 - 172C + C^2 = 7056 - 168C + C^2$ and so $144 - 4C = 0$ and we have $C = 36$.

6: (a) A tank initially contains 20 L of pure water. Brine containing 5 grams of salt per liter of water enters the tank at 6 L/min. The solution is kept well mixed and drains from the tank at 2 L/min. Find the concentration of salt in the tank when the tank contains 80 L of brine.

Solution: Let $S(t)$ be the amount of salt in the tank (in grams) at time t (in minutes), and let $V(t)$ be the volume of brine in the tank (in litres). Note that $S(0)=0$ and $V(0)=20$. Also, let r_{in} and r_{out} be the incoming and outgoing rates, and let c_{in} and c_{out} be the incoming and outgoing concentrations. We have $r_{in}=6$, $r_{out}=2$, $c_{in}=5$ and $c_{out}=S(t)/V(t)$.

The volume $V(t)$ satisfies the IVP $V'=r_{in}-r_{out}=6-2=4$ with $V(0)=20$, and the solution is easily found to be $V(t)=20+4t$. The amount of salt $S(t)$ then satisfies the IVP

$$S'=r_{in}c_{in}-r_{out}-c_{out}=6\cdot 5-2\,\frac{S}{20+4t}=30-\frac{S}{10+2t}$$

with $S(0)=0$. The DE is linear since we can write it as $S'+\dfrac{1}{2(5+t)}\,S=30$. An integrating factor is $\lambda=e^{\int\frac{1}{2(5+t)}\,dt}=e^{\frac{1}{2}\ln(5+t)}=(5+t)^{1/2}$, and the general solution is

$$S(t)=(5+t)^{-1/2}\int 30\,(5+t)^{1/2}\,dt=(5+t)^{-1/2}\big(20\,(5+t)^{3/2}+c\big)=20\,(5+t)+\frac{c}{\sqrt{5+t}}\,.$$

Since $S(0)=0$ we have $100+c/\sqrt{5}=0$ and so $c=-100\sqrt{5}$ and the solution is

$$S(t)=20\,(5+t)-\frac{100\sqrt{5}}{\sqrt{5+t}}\,.$$

The tank contains 80 litres when $V(t)=80$, that is when $20+4t=80$, so $t=15$. At $t=15$, the amount of salt is $S(15)=20\cdot 20-\frac{100\sqrt{5}}{\sqrt{20}}=400-50=350$, and the concentration of salt is $\dfrac{S(15)}{V(15)}=\dfrac{350}{80}=\dfrac{35}{8}$.

(b) A tank, in the shape of a lower-hemisphere of radius 1 m, is initially filled with water. Water drains through a circular hole of diameter 5 cm at the bottom of the tank. When the depth of water in the tank is equal to y m, the water flows through the hole at a speed of $4\sqrt{y}$ m/s. Determine the time it takes for the depth of the water in the tank to reach 25 cm.

Solution: The front view of the tank is shaped like the bottom half of the circle $x^2+(y-1)^2=1$, and the right half of this circle is given by $x=\sqrt{1-(y-1)^2}=\sqrt{2y-y^2}$. The horizontal cross-section of the tank at height y is a circle of radius $r=x=\sqrt{2y-y^2}$, and so a slice of thickness Δy has volume $\Delta V\cong\pi(2y-y^2)\Delta y$. Thus for a small time interval Δt we have $\frac{\Delta V}{\Delta t}\cong\pi(2y-y^2)\frac{\Delta y}{\Delta t}$. As $\Delta t\to 0$ we get

$$V'=\pi(2y-y^2)\,y'\,.$$

On the other hand, since the water flows through a hole of area $a=\pi(.025)^2=\pi\left(\frac{1}{40}\right)^2=\frac{\pi}{1600}$ m^2 at a speed $v=4\sqrt{y}$ m/s, we also have

$$V'=-av=-\tfrac{\pi}{400}\sqrt{y}\,.$$

Equating these two expressions for V' we obtain

$$\pi(2y-y^2)\,y'=-\tfrac{\pi}{400}\sqrt{y}\,.$$

This DE is separable as we can write it as $(2y^{1/2}-y^{3/2})y'=\frac{-1}{400}$. Integrate both sides, using the substitution $y=y(t)$ on the left, to get $\displaystyle\int 2y^{1/2}-y^{3/2}\;dy=\int -\tfrac{1}{400}\,dt$ which gives

$$\tfrac{4}{3}y^{3/2}-\tfrac{2}{5}\,y^{5/2}=-\tfrac{1}{400}\,t+c\,.$$

To get $y(0)=1$ we need $\frac{4}{3}-\frac{2}{5}=c$ so $c=\frac{14}{15}$ and so we have $\frac{4}{3}\,y^{3/2}-\frac{2}{5}\,y^{5/2}=-\frac{1}{400}\,t+\frac{14}{15}$. Thus we have $y(t)=\frac{1}{4}\iff\frac{1}{400}\,t=\frac{14}{15}-\frac{4}{3}\cdot\frac{1}{8}+\frac{2}{5}\cdot\frac{1}{32}=\frac{14}{15}-\frac{1}{6}+\frac{1}{80}\iff t=\frac{1120}{3}-\frac{200}{3}+5=\frac{935}{3}$.

7: (a) In a chemical reaction, 2 g of substance A reacts with 1 g of substance B to produce 3 g of substance C. Suppose that 4 g of substance A and 3 g of substance B are combined at time $t = 0$ min. Let $a(t)$, $b(t)$ and $c(t)$ be the amounts, in grams, of the three substances, and suppose that

$$c'(t) = 3\,a(t)\,b(t)\,.$$

Find a formula for $c(t)$, and find the time at which 3 g of substance C has been produced.

Solution: To produce c g of substance C we must use up $\frac{2}{3}\,c$ g of substance A and $\frac{1}{3}\,c$ g of substance B, and so $a(t) = 4 - \frac{2}{3}\,c(t)$ and $b(t) = 3 - \frac{1}{3}\,c(t)$. Thus $c' = 3ab = 3\left(4 - \frac{2}{3}\,c\right)\left(3 - \frac{1}{3}\,c\right) = (12 - 2c)\left(3 - \frac{1}{3}\,c\right)$ so

$$3\,c' = 2(6-c)(9-c)\,.$$

This DE is separable since we can write it as $\dfrac{3}{(6-c)(9-c)}\,c' = 2$. Integrate both sides to get

$$\int \frac{3\,dc}{(6-c)(9-c)} = \int 2\,dt = 2t + d\,.$$

We have $\displaystyle\int \frac{3\,dc}{(6-c)(9-c)} = \int \frac{1}{6-c} - \frac{1}{9-c}\;dc = -\ln(6-c) + \ln(9-c) + \text{const} = \ln\frac{9-c}{6-c} + \text{const}$, and so we have $\ln\dfrac{9-c}{6-c} = 2t + d$. To get $c(0) = 0$ we need $\ln\frac{3}{2} = d$ and so

$$\ln\frac{9-c}{6-c} = 2t + \ln\tfrac{3}{2} \quad (1)$$

Thus $\dfrac{9-c}{6-c} = e^{2t + \ln\frac{3}{2}} = \frac{3}{2}\,e^{2t} \implies 18 - 2c = 18e^{2t} - 3ce^{2t} \implies c(3e^{2t} - 2) = 18e^{2t} - 18 \implies c = \dfrac{18(e^{2t}-1)}{3e^{2t}-2}$
Finally, put $c = 3$ into equation (1) to get $\ln 2 = 2t + \ln\frac{3}{2} \implies 2t = \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3} \implies t = \frac{1}{2}\ln\frac{4}{3}\,.$

(b) Let $x(t)$ be the height of an object of mass m which is thrown upwards from the ground. If the force of air resistance is $-kx'$, then $x(t)$ satisfies the DE $mx'' + kx' + mg = 0$. Suppose that $m = 1$, $k = \frac{1}{10}$, $g = 10$, $x(0) = 0$ and $x'(0) = 20$. Find the time t at which the object reaches its maximum height, find $x(t)$, and determine (with the help of a calculator) whether the object takes longer on the way up to its maximum height or on the way back down to the ground.

Solution: Put in the given values for m, k and g to get $x'' + \frac{1}{10}x' + 10 = 0$. This is a linear DE for $v = x'$ since we can write it as $v' + \frac{1}{10}v = -10$. An integrating factor is $\lambda = e^{\int \frac{1}{10}\,dt} = e^{t/10}$, and the general solution is

$$v(t) = e^{-t/10}\int -10\,e^{t/10}\,dt = e^{-t/10}\big(-100\,e^{t/10} + c_1\big) = c_1\,e^{-t/10} - 100\,.$$

Put in $v(0) = x'(0) = 20$ to get $c_1 - 100 = 20$, so $c_1 = 120$ and we have

$$v(t) = 120\,e^{-t/10} - 100\,.$$

It reaches its maximum height when $v(t) = 0$, and we have $v(t) = 0 \implies 120\,e^{-t/10} - 100 = 0 \implies e^{-t/10} = \frac{100}{120} = \frac{5}{6} \implies e^{t/10} = \frac{6}{5} \implies \frac{1}{10}\,t = \ln\left(\frac{6}{5}\right) \implies t = 10\ln\left(\frac{6}{5}\right)$.

We have $x(t) = \displaystyle\int v(t)\,dt = \int 120\,e^{-t/10} - 100\;dt = -1200\,e^{-t/10} - 100\,t + c_2$. Put in $x(0) = 0$ to get $-1200 + c_2 = 0$, so $c_2 = 1200$ and we have

$$x(t) = -1200\,e^{-t/10} - 100\,t + 1200 = 1200\big(1 - e^{-t/10}\big) - 100\,t\,.$$

By part (c), it gets to the top at $t_1 = 10\ln\left(\frac{6}{5}\right)$. Consider its position at $t_2 = 2t_1 = 20\ln\left(\frac{6}{5}\right)$. If it takes longer on the way up, then it will land before $t = t_2$ and then $x(t_2) < 0$. If it takes longer on the way back down, then it will not yet have landed when $t = t_2$ and so we will have $x(t_2) > 0$. We have

$$x(t_2) = 1200\big(1 - e^{-2\ln(6/5)}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 1200\big(1 - \tfrac{25}{36}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 100\big(\tfrac{11}{3} - 20\ln\left(\tfrac{6}{5}\right)\big)\,.$$

A calculator shows that $20\ln\left(\frac{6}{5}\right) \cong 3.64 < \frac{11}{3}$, so $x(t_2) > 0$, and so it takes longer on the way back down.

8: (a) A **Bernoulli** DE is a DE which can be written in the form $y' + py = qy^n$ for some continuous functions p and q and some integer n. Show that the substitution $u = y^{1-n}$ transforms the above Bernoulli DE for $y = y(x)$ into a linear DE for $u = u(x)$.

Solution: Let $u = y^{1-n}$ so $u' = (1-n)\,y^{-n}\,y'$. Multiply both sides of the DE $y' + p\,y = q\,y^n$ by $(1-n)\,y^{-n}$ to get $(1-n)\,y^{-n}\,y' + p\,(1-n)\,y^{1-n} = q\,(1-n)$ which we can write as

$$u' + p(1-n)\,u = q(1-n)\,.$$

This is a linear DE for $u = u(x)$.

(b) Solve the IVP $y' + y = x\,y^3$, with $y(0) = 2$.

Solution: Let $u = y^{-2}$ so $u' = -2y^{-3}\,y'$, and multiply both sides of the DE $y' + y = x\,y^3$ by $-2y^{-3}$ to get $-2y^{-3}\,y' - 2y^{-2} = -2x$, that is

$$u' - 2u = -2x\,.$$

This is a linear DE for $u = u(x)$. An integrating factor is $I = e^{\int -2\,dx} = e^{-2x}$, and the general solution is $u = e^{2x}\displaystyle\int -2x\,e^{-2x}\,dx$. Integrate by parts using $u = x$, $du = dx$, $v = e^{-2x}$ and $dv = -2e^{-2x}\,dx$ to get

$$u = e^{2x}\left(x\,e^{-2x} - \int e^{-2x}\,dx\right) = e^{2x}\left(x\,e^{-2x} + \tfrac{1}{2}\,e^{-2x} + c\right) = x + \tfrac{1}{2} + c\,e^{2x}\,,$$

that is $y^{-2} = x + \frac{1}{2} + c\,e^{2x}$. To get $y(0) = 2$ we need $\frac{1}{4} = \frac{1}{2} + c$ so $c = -\frac{1}{4}$ and so we have

$$y^{-2} = x + \tfrac{1}{2} - \tfrac{1}{4}\,e^{2x} \Longrightarrow y = \left(x + \tfrac{1}{2} - \tfrac{1}{4}\,e^{2x}\right)^{-1/2} = \frac{2}{\sqrt{4x + 2 - e^{2x}}}\,.$$

(c) A **homogeneous** DE is a DE which can be written in the form $y' = F\left(\frac{y}{x}\right)$ for some continuous function F. Show that the substitution $u = \frac{y}{x}$ transforms a homogeneous DE for $y = y(x)$ into a separable DE for $u = u(x)$.

Solution: Let $u = \frac{y}{x}$, so $y = xu$. Then $y' = u + xu'$, so we can write the DE $y' = F\left(\frac{y}{x}\right)$ as $u + xu' = F(u)$. This is separable since we can write it as

$$\frac{u'}{F(u) - u} = \frac{1}{x}\,.$$

(d) Solve the IVP $y' = \dfrac{x^2 + 3y^2}{2xy}$ with $y(1) = 2$.

Solution: This DE is homogeneous since we can write it as $y' = \dfrac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$. Let $u = \frac{y}{x}$ so $y = xu$ and $y' = u + xu'$. Then we can write the DE as $u + xu' = \dfrac{1 + 3u^2}{2u}$, that is $xu' = \dfrac{1 + 3u^2}{2u} - u = \dfrac{1 + u^2}{2u}$. This is separable, as we can write it as $\dfrac{2u\,du}{1 + u^2} = \dfrac{dx}{x}$. Integrate both sides to get

$$\ln(1 + u^2) = \ln|x| + c \Longrightarrow 1 + u^2 = ax \text{ (where } a = \pm e^c) \Longrightarrow u = \pm\sqrt{ax - 1}$$
$$\Longrightarrow \frac{y}{x} = \pm\sqrt{ax - 1} \Longrightarrow y = \pm x\sqrt{ax - 1}\,.$$

To get $y(1) = 2$, we need $2 = \pm\sqrt{a - 1}$, so we need to use the $+$ sign and we need $a - 1 = 4$ so $a = 5$. Thus

$$y = x\sqrt{5x - 1}\,.$$

9: (a) The substitution $u(x) = y'(x)$ and $u'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', x)$ for $y = y(x)$ to a first order DE for $u = u(x)$. Use this substitution to solve the IVP $y'' - 2y' = 4x$ with $y(0) = 0$ and $y'(0) = 0$.

Solution: When we let $u = y'$ so that $u' = y''$, the DE becomes $u' - 2u = 4x$, which is linear. An integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$ and so the solution is $y' = u = e^{2x} \int 4x\,e^{-2x}\,dx$. We integrate by parts using $u = 4x$ and $dv = e^{-2x}\,dx$ so that $du = 4\,dx$ and $v = -\frac{1}{2}\,e^{-2x}$ to get

$$y' = e^{2x} \int 4x\,e^{-2x}\,dx = e^{2x}\left(-2x\,e^{2x} + \int 2\,e^{-2x}\,dx\right) = e^{2x}\left(-2x\,e^{-2x} - e^{-2x} + c_1\right) = c_1\,e^{2x} - 2x - 1\,.$$

Put in $y'(0) = 0$ to get $0 = c_1 - 1$ so that $c_1 = 1$, and so we have $y' = e^{2x} - 2x - 1$. Now integrate again to get

$$y = \int e^{2x} - 2x - 1\ dx = \tfrac{1}{2}\,e^{2x} - x^2 - x + c_2\,.$$

Put in $y(0) = 0$ to get $0 = \frac{1}{2} + c_2$, so we have $c_2 = -\frac{1}{2}$, and the solution is $y = \frac{1}{2}\,e^{2x} - x^2 - x - \frac{1}{2}$.

(b) The substitution $u(y(x)) = y'(x)$ and $u'(y(x))y'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', y)$ for $y = y(x)$ to a first order DE for $u = u(y)$. Use this substitution to solve the IVP $y\,y'' + (y')^2 = 0$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: Let $u(y(x)) = y'(x)$ so $u'(y(x))y'(x) = y''(x)$, that is $y' = u$ and $y'' = u\,u'$. The DE becomes $yu\,u' + u^2 = 0$. This is linear since we can write it as $u' + \frac{1}{y}u = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$ and the solution is $u = \frac{1}{y}\int 0\,dy = \frac{a}{y}$. Put in $x = 1$, $y = 2$, $u = y' = 3$ to get $3 = \frac{a}{2}$ so $a = 6$ and the solution is $u = \frac{6}{y}$, that is $y' = \frac{6}{y}$. This DE is separable since we can write it as $y\,y' = 6$. Integrate both sides (with respect to x) to get $\frac{1}{2}y^2 = 6x + c$. Put in $x = 1$, $y = 2$ to get $2 = 6 + x$ so $c = -4$ and the solution is $\frac{1}{2}y^2 = 6x - 4$, that is $y = \pm\sqrt{12x - 8}$. Since $y(1) = 2$, we must use the $+$ sign, so $y = \sqrt{12x - 8}$.

(c) Solve the IVP $y'' + (y')^2 = 2e^{-y}$ with $y(0) = 0$ and $y'(0) = 2$.

Solution: Let $u(y(x)) = y'(x)$ so $u'(y(x))y'(x) = y''(x)$, that is $y' = u$ and $y'' = u\,u'$. Then we can write the given DE as $u\,u' + u^2 = 2e^{-y}$, that is

$$u' + u = 2e^{-y}\,u^{-1}\,.$$

This is a Bernoulli equation. Let $v = u^2$ so $v' = 2u\,u'$. Multiply the Bernoulli equation by $2u$ to get $2u\,u' + 2u^2 = 4e^{-y}$, and write this as

$$v' + 2v = 4e^{-y}\,.$$

This is linear. An integrating factor is $\lambda = e^{\int 2\,dy} = e^{2y}$ and the solution is

$$v = e^{-2y}\int 4e^{y} = e^{-2y}(4e^{y} + b)\,.$$

From the initial conditions, when $x = 0$ we need $y = 0$, and $u = y' = 2$ so $v = u^2 = 4$, and so $4 = 4 + b$, that is $b = 0$. Thus we have

$$v = 4e^{-y}\,.$$

Since $v = u^2 = (y')^2$, we have $(y')^2 = 4e^{-y}$ so $y' = \pm 2e^{-y/2}$. Since $y'(0) = 2$ we must use the $+$ sign, so

$$y' = 2e^{-y/2}\,.$$

This DE is separable. We write it as $e^{y/2}dy = 2dx$ and integrate both sides to get

$$2e^{y/2} = 2x + c\,.$$

To get $y(0) = 0$ we need $2 = c$, so the solution is given by $2e^{y/2} = 2x + 2$. Solve for $y = y(x)$ to get

$$y = 2\ln(x + 1)\,.$$

10: An object of mass m falls towards the Earth. The force due to gravity is $F = -\dfrac{GMm}{x^2}$, where x is the distance from the center of the Earth to the object, G is the gravitational constant and M is the mass of the Earth.

(a) If $x(0) = x_0$ and $x'(0) = 0$ then find the velocity x' as a function of x.

Solution: We have $F = -\dfrac{GMm}{x^2}$ and $F = ma = mx''$, and so $x(t)$ satisfies the DE

$$x'' = -\frac{GM}{x^2}\ , \text{ with } x(0) = 0 \text{ and } x'(0) = x_0\,.$$

The independent variable t does not occur explicitly in the DE, so we let $x' = v$ and $x'' = v\,v'$ where $v' = \frac{dv}{dx}$. The DE becomes $v\,v' = -\dfrac{GM}{x^2}$. This DE is separable, so we write it as $v\,dv = -\dfrac{GM}{x^2}\,dx$ and integrate both sides to get $\frac{1}{2}\,v^2 = \dfrac{GM}{x} + c_1$. Put $x = x_0$ and $u = 0$ to get $c_1 = -\dfrac{GM}{x_0}$, and so we have $\frac{1}{2}\,v^2 = GM\left(\frac{1}{x} - \frac{1}{x_0}\right)$, that is $v = \pm 2\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. We are interested in the case that $v = x' \le 0$, so

$$v = -\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}\,.$$

We remark that (if you know some physics) this formula can also be obtained using conservation of energy.

(b) Find the time t as a function of x, and then find the time at which $x = \frac{1}{2}\,x_0$.

Solution: Replace v by x' again to get $x' = -\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. This DE is separable, so we write it as $\dfrac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\sqrt{2GM}\,dt$ and integrate both sides to get $\displaystyle\int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\int \sqrt{2GM}\,dt = -\sqrt{2GM}\,t + c_2$. Let I be the integral on the left. Then

$$I = \int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = \int \frac{\sqrt{x}\,dx}{\sqrt{1 - \frac{x}{x_0}}} = \int \frac{2\,x_0\sqrt{x_0}\,u^2}{\sqrt{1-u^2}}\,du\,,$$

where $u^2 = \dfrac{x}{x_0}$ so $\sqrt{x} = \sqrt{x_0}\,u$ and $2\,x_0\,u\,du = dx$. Now let $\cos\theta = u$ so that $\sin\theta = \sqrt{1-u^2}$ and $-\sin\theta\,d\theta = dx$. Then

$$\begin{aligned} I &= -\int 2\,x_0\sqrt{x_0}\,\cos^2\theta\,d\theta = -x_0\sqrt{x_0}\big(\theta + \sin\theta\cos\theta\big) + c_3 = -x_0\sqrt{x_0}\big(\cos^{-1}u + u\sqrt{1-u^2}\big) + c_3 \\ &= -x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) + c_3\,. \end{aligned}$$

Since $I = -\sqrt{2GM}\,t + c_2$, we obtain

$$-x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) = -\sqrt{2GM}\,t + c\,.$$

Put in $t = 0$ and $x = x_0$ to get $c = 0$, and so we have

$$t = \frac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0} - \left(\frac{x}{x_0}\right)^2}\,\right).$$

Finally, when $x = \frac{1}{2}\,x_0$ we have $t = \dfrac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \dfrac{x_0\sqrt{x_0}\,(\pi + 2)}{4\sqrt{2GM}}$.