MATH 148 Calculus 2, Exercises for Chapter 5

1: (a) Verify that $y = x \sin x$ is a solution of the DE $y\,(y'' + y) = x \sin 2x$.

(b) Find all the solutions of the form $y = ax^2 + bx + c$ to the DE $\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1$.

(c) Find constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

2: Find the general solution to each of the following DEs.

(a) $x\,y' + y = \sqrt{x}$

(b) $\sqrt{x}\,y' = 1 + y^2$

(c) $y' = 2xy^2 + y^2 + 8x + 4$

(d) $y' + y \tan x = \sin^2 x$

3: Find the solution to each of the following IVPs.

(a) $x\,y' = y^2 + y$ with $y(1) = 1$.

(b) $x\,y' + 2y = \ln x$ with $y(1) = 0$.

(c) $y' + xy = x^3$ with $y(0) = 1$.

(d) $y' = \dfrac{y}{x + y^2}$ with $y(3) = 1$.

4: Consider the IVP $y' = 2(x + y) - \frac{1}{2}$ with $y(0) = 0$.

(a) Find the exact solution $y = f(x)$ to the above IVP.

(b) Apply Euler's method with step size $\Delta x = \frac{1}{2}$ to find a polygonal approximation $y = g(x)$ for $0 \le x \le 2$ to the above solution $y = f(x)$.

(c) Sketch the direction field for the given DE along with the graph of the exact solution $y = f(x)$ and the graph of the polygonal solution $y = g(x)$.

(d) Let $g_n(x)$ be the polygonal approximation to the solution $y = f(x)$ obtained by applying Euler's method with step size $\Delta x = \frac{1}{n}$. Show that $\lim\limits_{n\to\infty} g_n(1) = f(1)$.

5: (a) The amount $A(t)$ of a radioactive substance satisfies the DE

$$A'(t) = k\,A(t)$$

for some constant $k < 0$. The substance has a half-life of 10 seconds, which means that $A(10) = \frac{1}{2}\,A(0)$. If $A(5) = 100$ then find the exact time t at which $A(t) = 20$.

(b) A pot of boiling water is removed from the heat and placed on a table in a room. The temperature $T(t)$ of the water at time t satisfies **Newton's Law of Cooling**, that is

$$T'(t) = k\big(C - T(t)\big)$$

for some constant $k > 0$, where C is the room temperature. After 2 minutes, the water has cooled from $100°$ to $84°$. After another 2 minutes, it has cooled to $72°$. What is the temperature in the room?

6: (a) A tank initially contains 20 L of pure water. Brine containing 5 grams of salt per liter of water enters the tank at 6 L/min. The solution is kept well mixed and drains from the tank at 2 L/min. Find the concentration of salt in the tank when the tank contains 80 L of brine.

(b) A tank, in the shape of a lower-hemisphere of radius 1 m, is initially filled with water. Water drains through a circular hole of diameter 5 cm at the bottom of the tank. When the depth of water in the tank is equal to y m, the water flows through the hole at a speed of $4\sqrt{y}$ m/s. Determine the time it takes for the depth of the water in the tank to reach 25 cm.

7: (a) In a chemical reaction, 2 g of substance A reacts with 1 g of substance B to produce 3 g of substance C. Suppose that 4 g of substance A and 3 g of substance B are combined at time $t = 0$ min. Let $a(t)$, $b(t)$ and $c(t)$ be the amounts, in grams, of the three substances, and suppose that

$$c'(t) = 3\,a(t)\,b(t)\,.$$

Find a formula for $c(t)$, and find the time at which 3 g of substance C has been produced.

(b) Let $x(t)$ be the height of an object of mass m which is thrown upwards from the ground. If the force of air resistance is $-kx'$, then $x(t)$ satisfies the DE $mx'' + kx' + mg = 0$. Suppose that $m = 1$, $k = \frac{1}{10}$, $g = 10$, $x(0) = 0$ and $x'(0) = 20$. Find the time t at which the object reaches its maximum height, find $x(t)$, and determine (with the help of a calculator) whether the object takes longer on the way up to its maximum height or on the way back down to the ground.

8: (a) A **Bernoulli** DE is a DE which can be written in the form $y' + py = qy^n$ for some continuous functions p and q and some integer n. Show that the substitution $u = y^{1-n}$ transforms the above Bernoulli DE for $y = y(x)$ into a linear DE for $u = u(x)$.

(b) Solve the IVP $y' + y = x\,y^3$, with $y(0) = 2$.

(c) A **homogeneous** DE is a DE which can be written in the form $y' = F\left(\frac{y}{x}\right)$ for some continuous function F. Show that the substitution $u = \frac{y}{x}$ transforms a homogeneous DE for $y = y(x)$ into a separable DE for $u = u(x)$.

(d) Solve the IVP $y' = \dfrac{x^2 + 3y^2}{2xy}$ with $y(1) = 2$.

9: (a) The substitution $u(x) = y'(x)$ and $u'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', x)$ for $y = y(x)$ to a first order DE for $u = u(x)$. Use this substitution to solve the IVP $y'' - 2y' = 4x$ with $y(0) = 0$ and $y'(0) = 0$.

(b) The substitution $u(y(x)) = y'(x)$ and $u'(y(x))y'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', y)$ for $y = y(x)$ to a first order DE for $u = u(y)$. Use this substitution to solve the IVP $y\,y'' + (y')^2 = 0$ with $y(1) = 2$ and $y'(1) = 3$.

(c) Solve the IVP $y'' + (y')^2 = 2e^{-y}$ with $y(0) = 0$ and $y'(0) = 2$.

10: An object of mass m falls towards the Earth. The force due to gravity is $F = -\dfrac{GMm}{x^2}$, where x is the distance from the center of the Earth to the object, G is the gravitational constant and M is the mass of the Earth.

(a) If $x(0) = x_0$ and $x'(0) = 0$ then find the velocity x' as a function of x.

(b) Find the time t as a function of x, and then find the time at which $x = \frac{1}{2}\,x_0$.